MATHEMATICS

JEE (Mains + Advanced)

Compound Angle

- Theory
- DPP
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- Answer Key

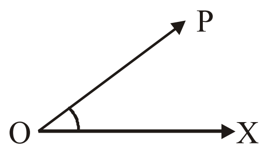
COMPOUND ANGLE

BASICS OF TRIGONOMETRY AND MEASUREMENT OF ANGLES

ANGLE

An angle can be thought of as the amount of rotation required to take one ray to another ray with a common point.

An angle XOP, is formed by the two rays OX and OP. The point O is called the 'vertex' and the half lines are called the sides of the angle.



An angle is generated by revolving a ray from the initial position OX to a terminal side OP. $\overrightarrow{OX}$ is called the initial side and $\overrightarrow{OP}$ is called the terminal side of the angle.

Angles are often labelled with Greek letters, for example θ. Sometimes an arrow is used to indicate the direction of the rotation.

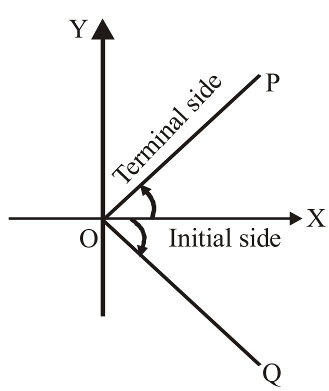
An angle is said to be **positive** if the direction of rotation is **anticlockwise** and **negative** if the direction of rotation is **clockwise**.

STANDARD POSITION OF AN ANGLE

In a rectangular cartesian (coordinate) system, the angle 'θ' is said to be in the **"standard position"**

1) If the vertex of an angle θ is at the origin and

2) If the initial side of the angle coincides with the positive x-axis. ($\overrightarrow{OX}$). Here $\angle XOP$ is positive where as $\angle QOX$ is negative.



MEASURES OF ANGLES

It has been seen in geometry that

1. When one line is perpendicular to another line each of the angles made at their point of intersection is a right angle.

2. All right angles are similar to each other.

In geometrical propositions angles are compared and one angle is shown to be greater or less than the other. But geometry, with the exception of a few cases does not show by exactly how much that one angle is greater or less than the other. In order to show this measurement is necessary; and in order to measure, a unit angle of measurement must be chosen. There are three types of measures to measure an angle

(i) Sexagesimal measure

(ii) Centesimal measure

(iii) Radian measure

SEXAGESIMAL MEASUREMENT (The British System)

A degree (°) is defined as the measure of the central angle subtended by an arc of a circle equal to 1/360 of the circumference of the circle or 1/90 of right angle. A minute (') is 1/60 of a degree and a second (") is 1/60 of a minute or 1/3600 of a degree.

One positive rotation = 360°

One degree = 1° = 60 minutes = 60' and

One minute = 1' = 60 seconds = 60"

Straight angle = 180°

Right angle = 90°

The magnitude of an angle containing 37 degrees and 42 minutes and 35 seconds is written as 37°42'35", read 37 degrees, 42 minutes, 35 seconds.

This system of measurements is sometimes called **the rectangular system** or **the sexagesimal system**.

CENTESIMAL MEASUREMENT (The French System)

A grade (^{g}) is defined as the measure of the central angle subtended by arc of a circle and is equal to 1/400 of the circumference of the circle. A minute (') is 1/100 of a grade and a second (") is 1/100 of a minute.

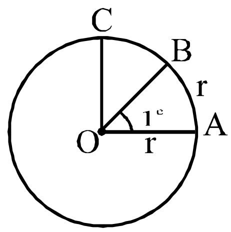
One Positive rotation	$= 400^g$
Straight angle	$= 200^g$
Right angle	$= 100^g$
$1^g = 100'$ and 1'	= 100 "

NOTE :- A second and a minute in the sexagesimal system and these in the centesimal system are not equal.

CIRCULAR (RADIAN) MEASURE

The radian is a natural unit measuring angles. We use radian in calculus because it makes the derivatives of trigonometric functions simple.

A radian (^{C}) is defined as the measure of the central angle subtended by an arc of a circle equal to the radius of the circle.



In above fig., O be the centre and 'r' be the radius of the circle.

$\angle AOB = 1$ radian, $\angle AOC = 1$ right angle.

Length of an arc AB = r

Length of an arc AC = $\frac{1}{4}(2\pi r) = \frac{\pi r}{2}$

Since arcs of the same circle are proportional to the angles subtended by them at the centre, we get $\frac{\angle AOB}{\angle AOC} = \frac{\text{arc AB}}{\text{arc AC}}$.

$$\Rightarrow \frac{1 \text{ radian}}{1 \text{ right angle}} = \frac{r}{\frac{\pi r}{2}} = \frac{2}{\pi}$$

$\Rightarrow$ π radian = 2 right angle

$\Rightarrow$ 1 radian = $\left(\frac{2}{\pi}\right)$ right angle = constant

$$1 \text{ radian} = \frac{180^o}{\pi} = (57.296)^o = 57^o 17' 45''$$

$$1 \text{ degree} = \frac{\pi}{180^o} \text{ radian} = 0.017453 \text{ radian.}$$

where $\pi \approx 3.14159$

The following equation will help us to convert the angle from one system to other.

$$\frac{180}{D} = \frac{200}{G} = \frac{\pi}{R} \left(\begin{array}{l}\text{Here D = degrees,} \\ \text{G = grades, R = radians}\end{array}\right)$$

(i) A radian is a constant angle.

(ii) 1 right angle = $\frac{\pi}{2}$ radians $= 90^o = 100^g$

(iii) 2 right angles = π radians = 180° = 200

E.g. Express 120^g in terms of radians.

$$100° = \frac{\pi}{2} \Rightarrow 1^g = \frac{\pi}{200}$$

$$\Rightarrow 120^g = 120\left(\frac{\pi}{200}\right) = \frac{3\pi}{5} \text{radians}$$

E.g. Express 150° in terms of radians.

$$1^o = \frac{\pi}{180} \Rightarrow 150^o = 150\left(\frac{\pi}{180}\right) = \frac{5\pi}{6} \text{ radians}$$

(iv) $\frac{\pi}{2} = 90^o, \frac{\pi}{3} = 60^o, \frac{\pi}{6} = 30^o, \frac{\pi}{4} = 45^o, \frac{3\pi}{2} = 270^o,$

$2\pi = 360°$ etc ;

(v) If no unit of measurement is indicated for an angle it will be understood that radian measure is implied.

(vi) If θ be the angle subtended by an arc of length 'l' on a circle of radius 'r', a its centre then length of the arc,

$$l = r\,\theta$$

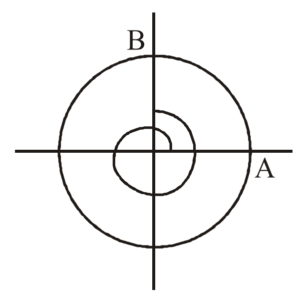
Area of the sector $= \frac{1}{2} lr = \frac{1}{2} r^2 \theta$

REAL NUMBERS AS RADIANS

Any real number can be though of as a radian measure, if we express the number as a multiple of 2π.

For example,

$\frac{5\pi}{2} = 2\pi \times \left(1 + \frac{1}{4}\right) = 2\pi + \frac{\pi}{2}$ corresponding to the arc length of $1\frac{1}{4}$ revolutions of the unit circle going anticlockwise from A to B.



Similarly, $27 \approx 4.297 \times 2\pi = 4 \times 2\pi + 0.297 \times 2\pi$

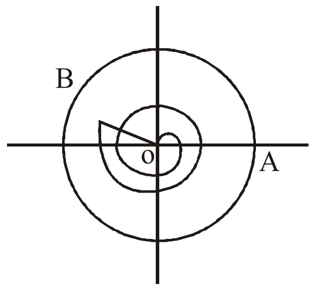
corresponding to an arc length of 4.297 revolutions of the unit going anticlockwise.

We can also think of negative numbers in terms of radians. Remember for negative radians we measure arc length clockwise around the unit circle.

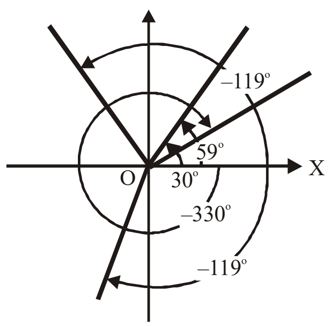
For example,

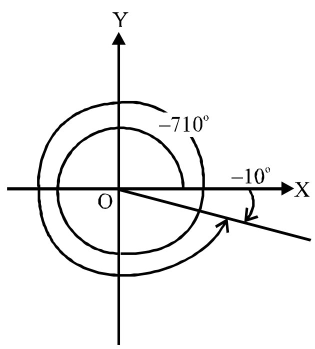
$-16 \approx -2.546 \times 2\pi = -2 \times 2\pi - (0.546) \times 2\pi$

corresponds to the arc length of approximately 2.546 revolutions of the unit circle going clockwise from A to B.



Angles in standard position





An angle is said to be a first quadrant angle or said to be in the first quadrant if in the standard position, its terminal side falls in that quadrant. Similar definitions hold for the other quadrants.

Coterminal angles

If two or more angles have the same terminal side in standard position, then they are called **coterminal angles**.

E.g. : 60°, 420°, –300° are coterminal.

Complementary Angles

Two angles are said to be **complementary** when their sum is π/2, i.e., the angles x,y are complementary angles off

$$x + y = \frac{\pi}{2}$$

E.g. 20°, 70° are complementary and $\frac{\pi}{12}, \frac{5\pi}{12}$ are complementary.

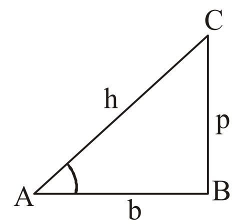
Supplementary Angles

Two angles are said to be supplementary when their sum is π.

E.g. 20°, 160° are supplementary and $\frac{\pi}{6}, \frac{5\pi}{6}$ are supplementary.

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

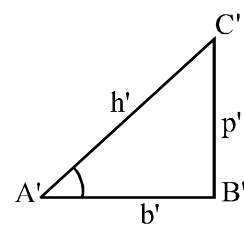
An angle whose measure is greater than 0° but less than 90° is called an acute angle. Consider a right-angled triangle ABC with right angle at B. The side opposite to the right angle is called the hypotenuse, the side opposite to angle A is called the perpendicular for angle A, and the side opposite to the third angle is called the base for angle A.



The ratio of any two sides of a triangle depends only on the measure of angle A. For example, consider a large and a small right-angled triangles as shown in fig. We have

$\frac{h}{h'} = \frac{b}{b'} = \frac{p}{p'}$, as these triangles are similar.

Thus, the ratio of the lengths of any two sides of a triangle is completely determined by angle A alone and is independent of the size of the triangle. There are six possible ratio that can be formed from the three sides of a right- angled triangle. Each of them has been given a name as follows :



(i) $\sin A = \frac{p}{h}$ (ii) $\cos A = \frac{b}{h}$

(iii) $\tan A = \frac{p}{b}$ (iv) $\cot A = \frac{b}{p}$

(v) $\sec A = \frac{h}{b}$ (vi) $\text{cosec}\, A = \frac{h}{p}$

The abbreviation sin, cos, tan, cot , sec, and cosec stand for sine, cosine, tangent, cotangent, secant, and cosecant of A, respectively. These functions of angle A are called trigonometric functions or trigonometrical ratios.

Trigonometric Identities

Trigonometric identities are equalities that involve trigonometric functions that are true for every single value of the occurring variables. In other words, they are equations that hold true regardless of the values of the angles being chosen.

Trigonometric identities are as follows :

$\sin^2\theta + \cos^2\theta = 1$, $\sec^2\theta - \tan^2\theta = 1$, $\theta \neq (2n+1)\dfrac{\pi}{2}$, $n \in I$

and $\text{cosec}^2\theta - \cot^2\theta = 1$, $\theta \neq n\pi$, $n \in I$

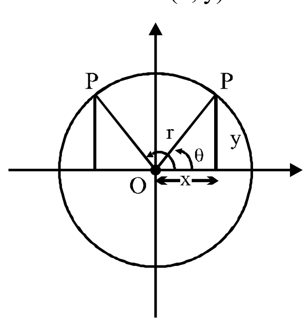
Trigonometric Ratios of Standard Angles

Angle(θ) T – ratio	30°	45°	60°
$\sin\theta$	$1/2$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$1/2$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
$\text{cosec}\,\theta$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

TRIGONOMETRIC FUNCTIONS

The definitions in the previous section apply to θ between 0 and $\dfrac{\pi}{2}$, since the angles in right angled triangle can never be greater than $\dfrac{\pi}{2}$. The definitions given below are useful in calculus, as they extend sin θ, cos θ and tan θ without restriction on the values of θ.

Consider a circle with center O and radius r in the rectangular cartesian coordinate system. Let θ be an angle (not quadrant) in standard position and let P(x, y) be any point, distinct from the origin, on the terminal side of the angle that intersects the circle in P(x, y).



∴ OP = r and $x^2 + y^2 = r^2$

The six trigonometric ratios of θ are defined, in terms of the abscissa, ordinate, and distance of P, as follows :

$\sin\theta = \dfrac{\text{ordinate}}{\text{distance}} = \dfrac{y}{r}$ $\qquad \cot\theta = \dfrac{\text{abscissa}}{\text{ordinate}} = \dfrac{x}{y}$

$\cos\theta = \dfrac{\text{abscissa}}{\text{distance}} = \dfrac{x}{r}$ $\qquad \sec\theta = \dfrac{\text{distance}}{\text{abscissa}} = \dfrac{r}{x}$

$\tan\theta = \dfrac{\text{ordinate}}{\text{abscissa}} = \dfrac{y}{x}$ $\qquad \csc\theta = \dfrac{\text{distance}}{\text{ordinate}} = \dfrac{r}{y}$

As an immediate consequence of these definitions, we have the so-called reciprocal relations :

$\sin\theta = \dfrac{1}{\csc\theta}$ $\qquad \cos\theta = \dfrac{1}{\sec\theta}$

$\tan\theta = \dfrac{1}{\cot\theta}$ $\qquad \cot\theta = \dfrac{1}{\tan\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$ $\qquad \csc\theta = \dfrac{1}{\sin\theta}$

Because of the reciprocal relationships, one ratio in each pair of reciprocal trigonometric ratios has been used more frequently than the other. The more frequently used trigonometric ratios are sine, cosine and tangent.

It is evident from the diagrams that the values of the trigonometric functions of θ changes as θ changes. The values of the functions of a given angle θ are independent of the choice of the point P on its terminal side.

NOTE

Since r is always positive, the signs of the ratios in the various quadrants depend on the signs of x and y.

The signs of the ratios sine, cosine and tangent in each of the quadrants

1. If $y > 0$ then $\sin\theta > 0$ and if $y < 0$ then $\sin\theta < 0$

$\Rightarrow$ sin θ is positive in Q_1, Q_2 and negative in Q_3, Q_4.

2. If $x > 0$ then $\cos\theta > 0 \Rightarrow$ cos θ is positive in Q_1 & Q_4.

3. If $x < 0$ then $\cos\theta < 0 \Rightarrow$ cos θ is negative in Q_2 & Q_3.

4. If $xy \geq 0$ then $\tan\theta > 0$ i.e., tan θ is positive in Q_1, Q_3.

5. If $xy \geq 0$ then $\tan\theta < 0$ i.e., tan θ is negative in Q_2 and Q_4.

Here Q_1 denotes with quadrant

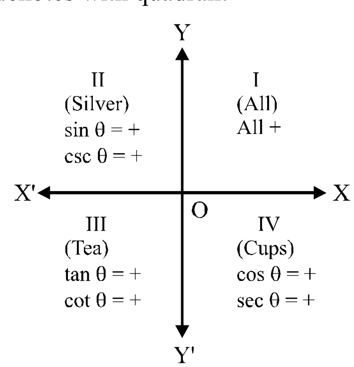


Illustration 1 Angle in 3rd quadrant whose sine and cosine are equal.

Solution In 3rd quadrant both sine and cosine angle are negative

$$\sin\left(\pi+\frac{\pi}{4}\right)=\cos\left(\pi+\frac{\pi}{4}\right),\text{ so }\frac{5\pi}{4}$$

Illustration 2 If $\sin\theta + \sin^2\theta = 1$, then find the value of

$$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1$$

Solution We have,

$\sin\theta + \sin^2\theta = 1$

$\Rightarrow \sin\theta = 1 - \sin^2\theta$

$\Rightarrow \sin\theta = \cos^2\theta$

Now,

$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1$

$= \cos^6\theta(\cos^6\theta + 3\cos^4\theta + 3\cos^2\theta + 1) - 1$

$= \cos^6\theta(\cos^2\theta + 1)^3 - 1$

$= \sin^3\theta(\sin\theta + 1)^3 - 1$

$= (\sin^2\theta + \sin\theta)^3 - 1$

$= 1 - 1$

$= 0$

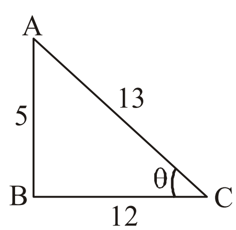
Illustration 3 If $\tan\theta + \sec\theta = 1.5$ find $\sin\theta$, $\tan\theta$, and $\sec\theta$.

Solution Given, $\sec\theta + \tan\theta = \frac{3}{2}$ .....(i)

Now, $\sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta} = \frac{2}{3}$ .....(ii)

Adding Eq. (i) and (ii),

we get $2\sec\theta = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$



$\therefore \quad \sec\theta = \frac{13}{12}$

$\therefore \quad \tan\theta = \frac{5}{12}$

and $\quad \sin\theta = \frac{5}{13}$

Illustration 4 If $\operatorname{cosec}\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, eliminate θ.

Solution Given, $\operatorname{cosec}\theta - \sin\theta = m$ or $\frac{1}{\sin\theta} - \sin\theta = m$

or $\frac{1-\sin^2\theta}{\sin\theta} = m$ or $\frac{\cos^2\theta}{\sin\theta} = m$ .....(i)

Again $\sec\theta - \cos\theta = n$ or $\frac{1}{\cos\theta} - \cos\theta = n$

or $\frac{1-\cos^2\theta}{\cos\theta} = n$ or $\frac{\sin^2\theta}{\cos\theta} = n$ .....(ii)

From Eq. (i), $\sin\theta = \frac{\cos^2\theta}{m}$ .....(iii)

Putting the value of $\sin\theta$ in eq. (ii), we get

$$\frac{\cos^4\theta}{m^2\cos\theta} = n \text{ or } \cos^3\theta = m^2n$$

$\therefore \quad \cos\theta = (m^2n)^{1/3}$ or $\cos^2\theta = (m^2n)^{2/3}$

From Eq. (iii),

$$\sin\theta = \frac{\cos^2\theta}{m} = \frac{(m^2n)^{2/3}}{m} = \frac{m^{4/3}n^{2/3}}{m}$$

$$= m^{1/3}n^{2/3} = (mn^2)^{1/3}$$

$\therefore \quad \sin^2\theta = (mn^2)^{2/3}$

Adding Eqs. (iv) and (v), we get

$(m^2n)^{2/3} + (mn^2)^{2/3} = \cos^2\theta + \sin^2\theta$

or $(m^2n)^{2/3} + (mn^2)^{2/3} = 1$

Illustration 5 Show that

$$2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$$

Solution : $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$

$= 2[(\sin^2 x)^3 + (\cos^2 x)^3] - 3(\sin^4 x + \cos^4 x) + 1$

$= 2[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)]$

$\quad - 3[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x] + 1$

$= 2[1 - 3\sin^2 x\cos^2 x] - 3[1 - 2\sin^2 x\cos^2 x] + 1 = 0$

Illustration 6 Prove that

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Solution L.H.S. $= \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}\frac{1+\sin\theta}{1+\sin\theta}}$

$$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \sec\theta + \tan\theta = \text{R.H.S.}$$

Illustration 7 Prove that

$$\frac{1}{\sec A-\tan A}-\frac{1}{\cos A}=\frac{1}{\cos A}-\frac{1}{\sec A+\tan A}$$

Solution Given, $\frac{1}{\sec A-\tan A}-\frac{1}{\cos A}=\frac{1}{\cos A}-\frac{1}{\sec A+\tan A}$

or $\quad \frac{1}{\sec A-\tan A}+\frac{1}{\sec A+\tan A}=\frac{1}{\cos A}+\frac{1}{\cos A}$

Here R.H.S. $=\frac{2}{\cos A}$

Now L.H.S. $=\frac{1}{\sec A-\tan A}+\frac{1}{\sec A+\tan A}$

$$=\frac{\sec A+\tan A+\sec A-\tan A}{(\sec A-\tan A)(\sec A+\tan A)}=\frac{2}{\cos A}$$

Thus, L.H.S. = R.H.S.

Illustration 8 If 3 sin θ + 5 cos θ = 5, then show that $5\sin\theta-3\cos\theta=\pm 3$.

Solution Given, $3\sin\theta+5\cos\theta=5$ .....(i)

Let $\quad 5\sin\theta-3\cos\theta=x$ .....(ii)

Squaring and adding, we get

$(9\sin^2\theta+25\cos^2\theta+30\sin\theta\cos\theta)$

$\quad +(25\sin^2\theta+9\cos^2\theta-30\sin\theta\cos\theta)=25+x^2$

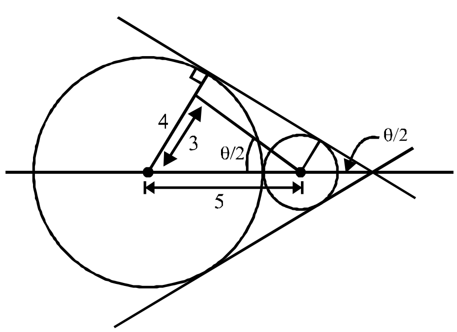
or $\quad 9(\sin^2\theta+\cos^2\theta)+25(\sin^2\theta+\cos^2\theta)=25+x^2$

or $\quad 34=25+x^2$ or $x^2=9$

or $\quad x=\pm 3$

Illustration 9 Two circles of radii 4 cm and 1 cm touch each other externally and θ is the angle contained by their direct common tangents. Find sin θ.

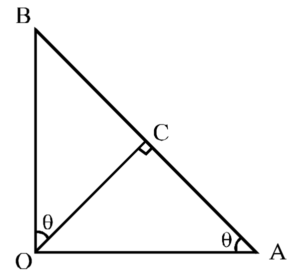
Solution From fig. we have,



$\Rightarrow \quad \sin\frac{\theta}{2}=\frac{3}{5} \qquad \Rightarrow \quad \cos\frac{\theta}{2}=\frac{4}{5}$

$\therefore \quad \sin\theta=2\times\frac{3}{5}\times\frac{4}{5}=\frac{24}{25}$

Illustration 10 If in fig. ∠BAO = $\tan^{-1}3$, then find the ratio BC : CA.



Solution From fig. we have

$\tan\theta=3$

$\therefore \quad \frac{OC}{AC}=\tan\theta,\ \frac{OC}{BC}=\cot\theta$

or $\quad \frac{BC}{AC}=\frac{\tan\theta}{\cot\theta}=\tan^2\theta=9$

Illustration 11 By geometrical interpretation, prove that

(i) $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$

(ii) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Solution Figure is self explanatory.

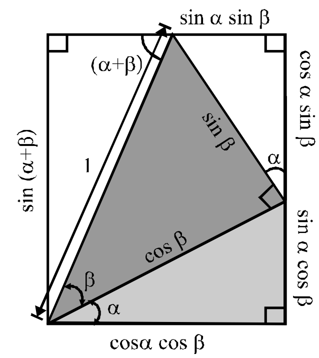
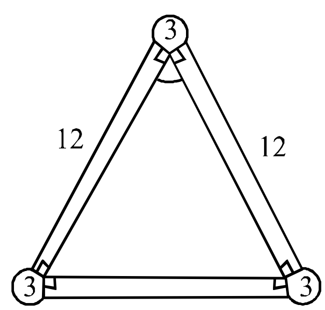
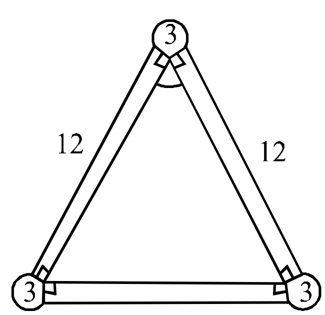


Illustration 12 A pool rack has 3 straight sides of length 12 cm each, rounded off by three circular arcs each of 3cms radius as shown in the figure. If the area inside the rack can be expressed as $a\pi+b\sqrt{3}+c$, where a, b, c are whole numbers, then the value of (a + b + c), is :



Solution The region bounded by the pool rack is as shown

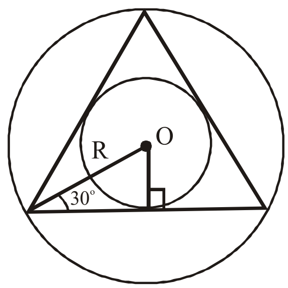


$$A = (12 \times 3)3 + 9\frac{1}{2}\left(\frac{2\pi}{3}\right)3 + \frac{\sqrt{3}}{4} \cdot 144$$

$$= 108 + 9\pi + 36\sqrt{3} = a\pi + b\sqrt{3} + c$$

$\therefore \quad (a + b + c) = 9 + 36 + 108 = 153.$

Illustration 13 The circumference of a circle circumscribing an equilateral triangle is 24π units. Find the area of the circle inscribed in the equilateral triangle.



Solution $2\pi R = 24\pi$ (R is the radius of circumcircle)

$R = 12$

$\sin 30^o = \frac{r}{R}$ (r is the radius of incircle)

$r = \frac{12}{2} = 6$

Area of incircle $= \pi r^2 = 36\pi$

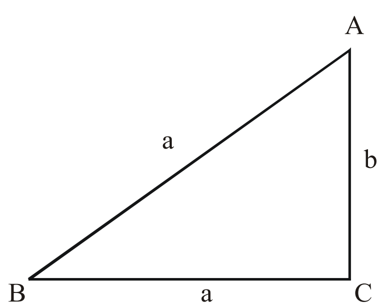
Illustration 14 If angle C of triangle ABC is 90°, then prove that

$\tan A + \tan B = \frac{c^2}{ab}$ (where, a, b, c, are sides opposite to angles A, B, C respectively).

Solution Draw ΔABC with $\angle C = 90^o$. We have

$$\tan A + \tan B = \frac{a}{b} + \frac{b}{a}$$

$$= \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}$$



DPP 1

Total Marks 40 **Time 30 Minute**

Question Number 1 to 10. **Marking Scheme :** +4 for correct answer 0 in all other cases.

1. Which of the following reduces to unity for $0° < A < 90°$

(i) $\sec^2 A - \sin^2 A \sec^2 A$

(ii) $\frac{1}{1+\sin^2 A} + \frac{1}{1+\operatorname{cosec}^2 A}$

(iii) $\frac{\tan^2 A \sin^2 A}{\tan^2 A - \sin^2 A}$

(iv) $\frac{\cot^2 A \cos^2 A}{\cot^2 A - \cos^2 A}$

(v) $(\sec^2 A - 1) \cot^2 A$

2. If $\sec\theta + \tan\theta = p$, then find the value of $\tan\theta$.

3. If $(1 + \sin A)(1 + \sin B)(1 + \sin C) = (1 - \sin A)(1 - \sin B)(1 - \sin C)$, then find the value of $(1 + \sin A)(1 + \sin B)(1 + \sin C)$.

4. If $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1, \frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta = 1,$ then eliminate θ.

5. If $a + b\tan\theta = \sec\theta$ and $b - a\tan\theta = 3\sec\theta$, then find the value of $a^2 + b^2$.

6. Express 45° 20'10" in radian measure ($\pi = 3.1415$).

7. Express 2.2 rad is degree measure.

8. Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15°.

9. Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an are of length 11cm.

10. If arcs of same length in two circles subtend angles of 60° and 75° at their centers, find the ratio of their radii.

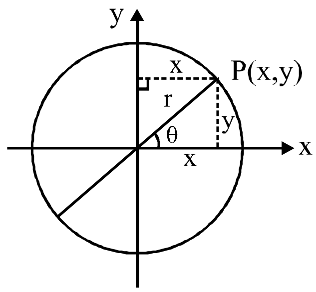
Result Analysis

1. 31 to 40 Marks : **Advance Level.**
2. 24 to 30 Marks : **Main Level.**
3. < 24 Marks : **Below Average**
(Please go through this article again)

REAL DEFINITION OF TWO BASIC FUNCTIONS (SINE & COSINE) - TRIGONOMETRIC RATIOS OF ANY MAGNITUDE - REDUCTION FORMULAE

A point P lies on circle such that it makes an angle θ with the x-axis. It is at a distance of x from y-axis and it a distance of y from x-axis and r is the radius of the circle.

Then sin θ = Distance of point P from x-axis radius of circle



$\therefore \quad \sin\theta = \dfrac{y}{r}$ and

$$\cos\theta = \frac{\text{Distance of point P from y-axis}}{\text{Radius of the circle}}$$

$\therefore \quad \cos\theta = \dfrac{x}{r}$

Limiting cases,

as $\theta \to 0, \quad y \to 0, \quad x = r$

$\therefore \quad \sin 0 = \dfrac{0}{r} = 0; \quad \cos 0 = \dfrac{r}{r} = 1$

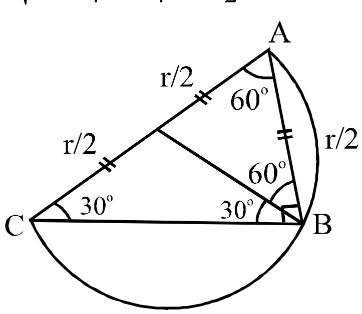
as $\theta \to 90°$ or $\theta = \dfrac{\pi}{2}$, $x = 0, y = r$

$\therefore \quad \sin 90° = \dfrac{r}{r} = 1; \quad \cos 90° = \dfrac{0}{r} = 0$

Now consider a triangle having angles 30°, 60°, 90°. Now taking AC as diameter construct a semi-circle passing through ABC. Now from B drop a bisector on AC.

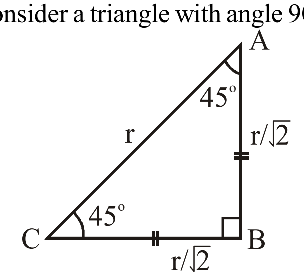
$\therefore \quad AC = r, \quad AB = \dfrac{r}{2}$

$\therefore \quad BC = \sqrt{r^2 - \dfrac{r^2}{4}} = \dfrac{\sqrt{3}r}{4} = \dfrac{\sqrt{3}r}{2}; \sin 30° = \dfrac{r}{2} \times \dfrac{1}{r} = \dfrac{1}{2};$



$$\cos 30° = \frac{\sqrt{3}r}{2r} = \frac{\sqrt{3}}{2}$$

Now consider a triangle with angle 90°, 45°, 45°



$\therefore$ AB = BC [Sides opposite to equal angles are equal]

$r^2 = 2(AB)^2 \Rightarrow AB = \dfrac{r}{\sqrt{2}}$

$\therefore \quad \sin 45° = \dfrac{r}{\sqrt{2}} \times \dfrac{1}{r} = \dfrac{1}{\sqrt{2}}; \quad \cos 45° = \dfrac{r}{\sqrt{2}} \times \dfrac{1}{r} = \dfrac{1}{\sqrt{2}}$

$\tan 45° = \cot 45° = 1$

Sine and Cosine of complementary angles are same

complementary angles are those whose sum is 90°

cos(90° – θ) = sin θ ;

sin(90° – θ) = cos θ ; tan (90° – θ) = cot θ

cot (90° – θ) = tan θ ;

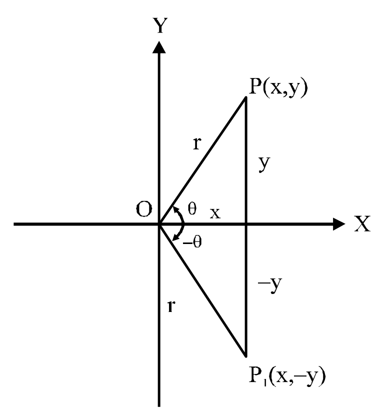
cosec (90° – θ) = sec θ ; sec (90° – θ) = cosec θ

NOTE : In first quadrant i.e., 0 to π/2 all trigonometrical ratios are positive.

REDUCTION FORMULAE

I. For all values of θ, the trigonometric ratios of an angle (–θ)

Let P_1 be the image of P in x-axis.



From figure, we have

$$\sin(-\theta) = -\frac{y}{r} = -\sin\theta$$

$$\cos(-\theta) = \frac{x}{r} = \cos\theta$$

$$\tan(-\theta) = -\frac{y}{x} = -\tan\theta \text{ and we have}$$

$\cot(-\theta) = -\cot\theta$

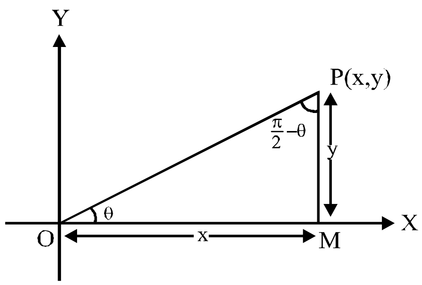
$\sec(-\theta) = \sec\theta$

$\csc(-\theta) = -\csc\theta$

E.g., $\sin(-50°) = -\sin 50°$, $\cos(-30°) = \cos 30°$, $\tan(-200°) = -\tan 200°$ etc.

II. For all values of θ, trigonometric ratios of the angle $\left(\frac{\pi}{2}-\theta\right)$ in terms of those of θ

Let θ be an angle in the standard position in the I[st] quadrant.



From ΔMOP

$$\sin\left(\frac{\pi}{2}-\theta\right) = \frac{x}{r} = \cos\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right) = \frac{y}{r} = \sin\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right) = \frac{x}{y} = \cot\theta$$

And we have also

$$\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$$

$$\sec\left(\frac{\pi}{2}-\theta\right) = \operatorname{cosec}\theta$$

$$\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right) = \sec\theta$$

III. (90 + θ) Relation :

Δ OPB and Δ P'O'B' are congruent by ASA property one ∠θ side, ∠(90° – θ).

∴ In Δ OP'B', P'B' =x as side opposite to 90° – θ is x in ΔOPB

in ΔOP'B', OB' = y as side opposite to θ in ΔOPB is y.

In ΔOP 'B'

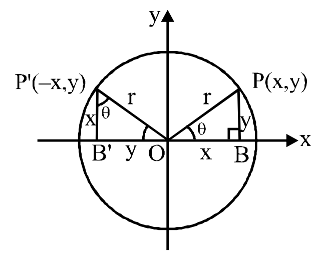
$$\sin(90° + \theta) = \frac{x}{r} = \cos\theta; \quad \cos(90° + \theta) = \frac{-y}{r} = -\sin\theta;$$

$\tan(90° + \theta) = -\cot\theta;$

$\cot(90° + \theta) = -\tan\theta;$

$\sec(90° + \theta) = -\operatorname{cosec}\theta;$

$\operatorname{cosec}(90° + \theta) = \sec\theta$



E.g., ∴ $\sin(120°) = \sin(90° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$

$\tan(135°) = \tan(90° + 45°) = -\cot 45° = -1$

$\cos(150°) = \cos(90° + 60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

IV. Reduction (180° – θ)

Δ's OPB and OP'B' are congruent by A S A.

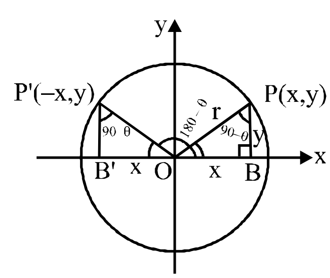
∠(90° – θ), side r, ∠θ

∴ side opposite to 90° – θ = x same as in ΔOPB and side opposite to θ = y same as in ΔOPB

$$\sin(180° - \theta) = \frac{y}{r} = \sin\theta; \quad \cos(180° - \theta)$$

$$= \frac{-x}{r} = -\cos\theta;$$

$\tan(180° - \theta) = -\tan\theta$; $\cot(180° - \theta) = -\cot\theta$;
$\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta$; $\sec(180° - \theta) = -\sec\theta$;
Sines of supplementary angles are equal supplementary angles
ar those whose sum is 180°.



Sum of the cosines, tangents, cotangents, secants of supplementary angles is zero.

since $\cos(180° - \theta) = -\cos\theta$

$\therefore \quad \cos(180° - \theta) + \cos\theta = 0$

same for tan, cot and sec

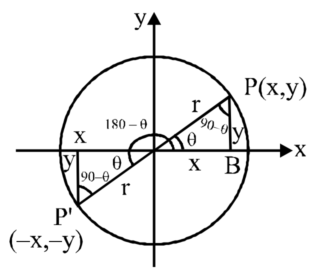
$\therefore \quad \sin(120°) = \sin(180° - 60°) = \sin 60° = \dfrac{\sqrt{3}}{2}$

$\cos(150°) = \cos(180° - 30°)$

$= -\cos 30° = -\dfrac{\sqrt{3}}{2}$

Sum of tangents of supplementary angles is zero.

V. Reduction (180° + θ)



ΔOPB and ΔOP'B' are congruent by ASA, Δ(90° – θ), side r, ∠θ.

$\therefore \quad \sin(180 + \theta) = \dfrac{-y}{r} = -\sin\theta;$

$\cos(180 + \theta) = \dfrac{-x}{r} = -\cos\theta;$

$\tan(180 + \theta) = \tan\theta; \cot(180 + \theta) = \cot\theta;$

$\text{cosec}(180 + \theta) = -\text{cosec}\,\theta;$

$\sec(180 + \theta) = -\sec\theta;$

E.g., $\sin(210°) = \sin(180° + 30°) = -\sin 30° = -1/2$

$\cos(240°) = \cos(180° + 60°) = -\cos 60° = -1/2$

$\tan(225°) = \tan(180° + 45°) = \tan 45° = 1$

$\sin(270°) = \sin(180° + 90°) = -\sin 90° = -1$

VI. Reduction (360 – θ) or (2π – θ)

Any angle of the form $2\pi - \theta$ can be written as $-\theta$ because if we say $2\pi - \theta$ then it means we are moving clockwise from origin and by convention all angles measured clockwise are –ve.

$\therefore \quad \sin(2\pi - \theta) = \sin(-\theta)$

$\cos(2\pi - \theta) = \cos(-\theta)$

$\tan(2\pi - \theta) = \tan(-\theta)$

Again ΔOPB and ΔOP'B are congruent by ASA $\sin(-\theta)$

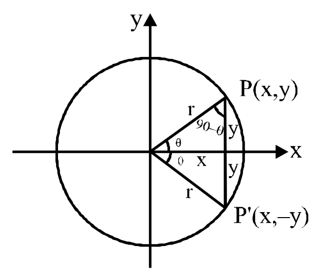
$= \dfrac{-y}{r} = -\sin\theta; \cos(-\theta) = \dfrac{x}{r} = \cos\theta;$

$\tan(-\theta) = -\tan\theta; \ \cot(-\theta) = -\cot\theta;$

$\text{cosec}(-\theta) = -\text{cosec}\,\theta; \sec(-\theta) = -\sec\theta;$

E.g., $\cos(315°) = \cos(360° - 45°) = \cos(-45°)$

$= \cos(45°) = \dfrac{1}{\sqrt{2}}$



$\tan(330°) = \tan(360° - 30°)$

$= \tan(-30°) = -\tan 30° = -\dfrac{1}{\sqrt{3}}$

$\tan(-120°) = -\tan 120° = -\tan(180° - 60°)$

$= \tan 60° = \sqrt{3} \ \cos(180°) = \cos(90° + 90°)$

$= \cos(180° - 0) = \cos(180 + 0°)$

$= -1 = -1 = -1$

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degree	0	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0
cot	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	ND

HIGHLIGHT OF THE TABLE

(1) $-1 \le \sin\theta \le 1$

(2) $-1 \le \cos\theta \le 1$

$\tan\theta$, $\cot\theta$ can attain all values from $(-\infty, \infty)$ i.e. any real value.

(3) $\sin\theta = 0, \sin\pi = 0, \sin 2\pi = 0$

$\Rightarrow$ $\sin n\pi = 0$ where $n \in I$;

$\therefore$ sine of integral multiple of π is zero.

(4) $\tan 0 = 0, \tan\pi = 0$

$\Rightarrow$ $\tan n\pi = 0$, where $n \in I$;

$\therefore$ tangent of integral multiple of π is zero.

(5) $\cos\frac{\pi}{2} = 0;\ \cos\frac{3\pi}{2} = 0;\ \cos-\frac{\pi}{2} = 0;$

$\Rightarrow \cos(2n-1)\frac{\pi}{2} = 0,$ where $n \in I$;

cosine of odd integral multiple of $\frac{\pi}{2}$ is zero.

(6) $\cot\frac{\pi}{2} = 0;\ \cot\frac{3\pi}{2} = 0;$

$\therefore \cos(2n-1)\frac{\pi}{2} = 0$, where $n \in 1$.

contangent of odd integral multiple of $\frac{\pi}{2}$ is zero.

(7) $\cos 0 = 1, \cos\pi = -1, \cos 2\pi = 0$

$\therefore \cos 2m\pi = 1; \cos(2m-1)\pi = -1$, where $m \in 1$

cosine of odd integral multiple of p is –1 and even integral multiple of π is 1. Same is the fate of sec.

$\sec 2n\pi = 1, \sec(2m-1)\pi = -1$

(8) $\tan\frac{\pi}{2} = \text{ND};\ \tan\frac{3\pi}{2} = \text{ND};\ \tan(2n+1)\frac{\pi}{2} = \text{ND},$

where $n \in I$ tangent of odd integral multiple of $\frac{\pi}{2}$ is ND

(9) $\cot 0° = \text{ND}, \cot\pi = \text{ND}, \cot 2\pi = \text{ND}$;

$\therefore$ $\cot n\pi =$ ND, where $n \in I$

cotangent of integral multiple of p is not defined.

(10) $\sin\left(2n\pi + \frac{\pi}{2}\right) = 1, \sin\left(2n\pi - \frac{\pi}{2}\right) = -1$

$= \sin\left(2n\pi + \frac{3\pi}{2}\right)$

Illustration 1 Find the value of following

(i) Find $\tan\left(-\frac{2\pi}{3}\right)$, cos315°, sec210°, cosec225°, cot(–300°) and $\sin\left(-\frac{17\pi}{4}\right)$.

(ii) Find the value of $\sin^2\frac{\pi}{4} + \sin^2\frac{3\pi}{4} + \sin^2\frac{5\pi}{4} + \sin^2\frac{7\pi}{4}$.

Solution : (i) $\tan\left(\frac{-2\pi}{3}\right) = -\tan\frac{2\pi}{3} = -\left(-\sqrt{3}\right) = \sqrt{3}$

$\cos 315° = \cos(360° - 45°) = \cos 45° = \frac{1}{\sqrt{2}}$

$\sec 210° = \sec(180° + 30°) = -\sec 30° = \frac{-2}{\sqrt{3}}$

$\text{cosec } 225° = \text{cosec}(180° + 45°) = -\text{cosec } 45°$

$= -\sqrt{2}$

$\cot(-300°) = -\cot(300°) = -\cot(360° - 60°)$

$= \cot 60° = \frac{1}{\sqrt{3}}$

$\sin\left(\frac{-17\pi}{4}\right) = -\sin\left(\frac{17\pi}{4}\right) = -\sin\left(4\pi + \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

(ii) $\sin^2\frac{\pi}{4} + \sin^2\frac{3\pi}{4} + \sin^2\frac{5\pi}{4} + \sin^2\frac{7\pi}{4}$

$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 2$

Illustration 2 If $x = \sin 1$, $y = \sin 1°$ then show that $x > y$.

Solution In a 1st quadrant, sin x increases from 0 to 1 when x increases from 0° to 90°.

We have $1 > 1°$ and 1, 1° are both in Q_1.

Hence, $\sin 1 > \sin 1°$

Illustration 3 Prove that sin(–420°) (cos 390°) + cos(–660°) sin(330°) = –1.

Solution L.H.S. = sin(–420°) (cos 390°) + cos (–660°) (sin 330°)

$= -\sin 420° \cos 390° + \cos 660° \sin 330°$

$[\because \sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta]$

$= \sin(90° \times 4 + 60°)\cos(90° \times 4 + 30°)$

$+ \cos(90° \times 7 + 30°)\sin(90° \times 3 + 60°)$

$= -(\sin 60°)(\cos 30°) + (\sin 30°)(-\cos 60°)$

$= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = -1 = \text{R.H.S.}$

Illustration 4 Prove that :

$$\frac{\cos(90^o+\theta)\sec(-\theta)\tan(180^o-\theta)}{\sec(360^o-\theta)\sin(180^o+\theta)\cot(90^o-\theta)}=-1$$

Solution L.H.S.= $\dfrac{\cos(90^o+\theta)\sec(-\theta)\tan(180^o-\theta)}{\sec(360^o-\theta)\sin(180^o+\theta)\cot(90^o-\theta)}$

$$=\frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sec\theta)(-\sin\theta)(\tan\theta)}=-1=\text{R.H.S.}$$

Illustration 5 If A,B,C,D are angles of a cyclic quadrilateral, then prove that,

$\cos A+\cos B+\cos C+\cos D=0$.

Solution We know that the opposite angles of a cyclic quadrilateral are supplementary,

i.e., $A+C=\pi$ and $B+D=\pi$. Therefore,

$A=\pi-C$ and $B=\pi-D$

$\Rightarrow\ \cos A=\cos(\pi-C)=-\cos C$

and $\cos B=\cos(\pi-D)=-\cos D$

$\therefore\ \cos A+\cos B+\cos C+\cos D$

$=-\cos C-\cos D+\cos C+\cos D=0$

Illustration 6. Show that $\tan 1^\circ \tan 2^\circ .... \tan 89^\circ = 1$.

Sol. L.H.S. $=(\tan 1^\circ\ \tan 89^\circ)(\tan 2^\circ \tan 88^\circ)$

$....(\tan 44^\circ \tan 46^\circ)\tan 45^\circ$

$=[\tan1^\circ \tan(90^\circ-1^\circ)]\,[\tan 2^\circ \tan(90^\circ-2^\circ)]$

$....[\tan 44^\circ \tan(90^\circ-44^\circ)]\tan 45^\circ$

$=(\tan 1^\circ \cot 1^\circ)(\tan 2^\circ \cot 2^\circ)....(\tan 44^\circ \cot 44^\circ)\tan 45^\circ=1$

= R.H.S. $[\because\ \tan\theta\cot\theta=1$ and $\tan 45^\circ=1]$

Illustration 7 Show that

$$\sin^2 5^\circ+\sin^2 10^\circ+\sin^2 15^\circ+....+\sin^2 90^\circ=9\frac{1}{2}.$$

Solution L.H.S. $=(\sin^2 5^\circ+\sin^2 85^\circ)+(\sin^2 10^\circ+\sin^2 80^\circ)$

$+....+(\sin^2 40^\circ+\sin^2 50^\circ)+\sin^2 45^\circ+\sin^2 90^\circ$

$=(\sin^2 5^\circ+\cos^2 5^\circ)+(\sin^2 10^\circ \cos^2 10^\circ)$

$+....+(\sin^2 40^\circ+\cos^2 40^\circ)+\sin^2 45^\circ+\sin^2 90^\circ$

$$(1+1+1+1+1+1+1+1)+\left(\frac{1}{\sqrt{2}}\right)^2+1$$

$$=9\frac{1}{2}=\text{R.H.S.}$$

Illustration 8 Find the value of

$$\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\frac{5\pi}{16}+\cos^2\frac{7\pi}{16}.$$

Solution L.H.S.

$$=\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\left(\frac{\pi}{2}-\frac{3\pi}{16}\right)+\cos^2\left(\frac{\pi}{2}-\frac{\pi}{16}\right)$$

$$=\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}+\sin^2\frac{\pi}{16}$$

$$=\left(\cos^2\frac{\pi}{16}+\sin^2\frac{\pi}{16}\right)+\left(\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}\right)$$

$=1+1=2.$

Illustration 9 Which of the following is possible ?

(A) $\sin\theta=\dfrac{5}{3}$ (B) $\tan\theta=1002$

(C) $\cos\theta=\dfrac{1+p^2}{1-p^2},(p\neq\pm1)$ (D) $\sec\theta=\dfrac{1}{2}$

Solution (B) $\sin\theta=\dfrac{5}{3}$ is not possible as $-1\le\sin\theta\le1$.

$\cos\theta=\dfrac{1+p^2}{1-p^2}$ is not possible, as in $\dfrac{1+p^2}{1-p^2}$, the numerator is always greater than the denominator for any value of p other than p = 0.

Hence, $\dfrac{1+p^2}{1-p^2}$ does not lie in [–1, 1].

$\sec\theta=\dfrac{1}{2}$ is not possible as $\sec\theta\in(-\infty,-1]\cup[1,\infty)$.

$\tan\theta=1002$ is possible as $\tan\theta$ can take any real value.

Illustration 10 Which of the following is greatest ?

(A) tan 1 (B) tan 4

(C) tan 7 (D) tan 10

Solution (A)

$\tan 4=\tan(\pi+(4-\pi))=\tan(4-\pi)=\tan(0.86)$

$\tan 7=\tan(2\pi+(7-2\pi))=\tan(7-2\pi)=\tan(0.72)$

$\tan 10=\tan(3\pi+(10-3\pi))$

$=\tan(10-3\pi)=\tan(10-3\pi)=\tan(0.58)$

Now, $1>0.86>0.72>0.58$

$\Rightarrow\ \tan 1>\tan(0.86)>\tan(0.72)>\tan(0.58)$ [as 1, 0.86, 0.72, 0.58 lie in the first quadrant and tangent functions increase in all the quadrant]. Hence, tan 1 is greatest.

Illustration 11 Which of the following is least ?

(A) sin 3 (B) sin 2

(C) sin 1 (D) sin 7

Solution (D)

$\sin 3 = \sin[\pi - (\pi - 3)] = \sin(\pi - 3) = \sin(0.14)$

$\sin 2 = \sin[\pi - (\pi - 2)]$

$= \sin(\pi - 2) = \sin(1.14)$

$\sin 7 = \sin[2\pi + (7 - 2\pi)] = \sin(0.72)$

Now $1.14 > 1 > 0.72 > 0.14$

$\Rightarrow$ $\sin(1.14) > \sin 1 > \sin(0.72) > \sin(0.14)$ [as 1.14, 1, 0.72, 0.14 lie in the first quadrant and sine functions increase in the first quadrant. Hence, sin 3 is least.

Illustration 12 If $A = 4\sin\theta + \cos^2\theta$, then which of the following is not true ?

(A) Maximum value of A is 5

(B) Minimum value of A is –4

(C) Maximum value of A occurs when $\sin\theta = 1/2$

(D) Minimum value of A occurs when $\sin\theta = 1$

Solution (A, C, D)

$f(\theta) = 4\sin\theta + \cos^2\theta = 4\sin\theta + 1 - \sin^2\theta$

$= 5 - (4 - 4\sin\theta + \sin^2\theta) = 5 - (\sin\theta - 2)^2$

Now maximum value of $f(\theta)$ occurs when

$(\sin\theta - 2)^2$ is minimum.

Minimum value of $(\sin\theta - 2)^2$ occurs when

$\sin\theta = 1$, then maximum value of $f(\theta)$ is $5 - (1 - 2)^2 = 4$.

Also minimum value of $f(\theta)$ occurs when $(\sin\theta - 2)^2$ is maximum.

Maximum value of $(\sin\theta - 2)^2$ occurs when $\sin\theta = -1$, then minimum value of $f(\theta)$ is $5 - (-1 - 2)^2 = -4$.

Illustration 13 Is the equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ possible for real values of x and y ?

Solution Given, $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$

Since $\sec^2\theta \geq 1$, we get $\dfrac{4xy}{(x+y)^2} \geq 1$

or $(x+y)^2 \leq 4xy$

or $(x+y)^2 - 4xy \leq 0$ or $(x-y)^2 \leq 0$

But for real values of x and y, $(x-y)^2 \geq 0$

Since $(x-y)^2 = 0$, $x = y$. Also $x + y \neq 0 \Rightarrow x \neq 0, y \neq 0$

Therefore, the given equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is possible for real values of x and y only when $x = y\ (x \neq 0)$.

Illustration 14 If $\operatorname{cosec}\theta = \dfrac{x^2 - y^2}{x^2 + y^2}$ where x, y are two unequal nonzero real numbers then prove that θ has no real value.

Solution Here $x \neq y$ and they are real, so $x^2 + y^2 > x^2 - y^2$

$\operatorname{cosec}\theta < 1$ if $x > y$ and $\operatorname{cosec}\theta > -1$ if $x < y$

$\therefore$ $-1 < \operatorname{cosec}\theta < 1$

Which is not possible because we know that $\operatorname{cosec}\theta \geq 1$ or ≤ -1

Hence, there is no real value of θ.

Illustration 15 Statement -1 : If $a, b, c \in R$ and not all equal, then

$\sec\theta = \dfrac{(bc + ca + ab)}{(a^2 + b^2 + c^2)}$ has a solution.

Statement - 2 : $\sec\theta \leq -1$ or $\sec\theta \geq 1$

Discuss above statements

Solution $(a - b)^2 > 0 \quad \Rightarrow \quad a^2 + b^2 > 2ab$

Similarly we have $a^2 + c^2 > 2ac$, $b^2 + c^2 > 2bc$ adding all inequalities, we get $ab + bc + ca < a^2 + b^2 + c^2$

$\Rightarrow \quad \dfrac{(bc + ca + ab)}{(a^2 + b^2 + c^2)} < 1$

Statement -1 is false and Statement -2 is true.

Illustration 16 Show that the equation $\sin\theta = x + \dfrac{1}{x}$ is impossible if x is real.

Solution Given, $\sin\theta = x + \dfrac{1}{x}$

$\therefore \quad \sin^2\theta = x^2 + \dfrac{1}{x^2} + 2x\dfrac{1}{x} = x^2 + \dfrac{1}{x^2} + 2$

$= \left(x - \dfrac{1}{x}\right)^2 + 4 \geq 4$

which is not possible since $\sin^2\theta \leq 1$.

Illustration 17 If $\sin^2\theta_1, + \sin^2\theta_2, + \sin^2\theta_3 = 0$, then which of the following is not the possible value of

$\cos\theta_1 + \cos\theta_2 + \cos\theta_3$.

(A) 3 (B) –3 (C) –1 (D) –2

Solution (D)

$\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 0$

$\Rightarrow \quad \sin^2\theta_1 = \sin^2\theta_2 = \sin^2\theta_3 = 0$

$\Rightarrow \quad \cos^2\theta_1, \cos^2\theta_2, \cos^2\theta_3 = 1$

$\Rightarrow \quad \cos\theta_1, \cos\theta_2, \cos\theta_3 = \pm 1$

$\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ can be –3 (when all are –1)

or 3 (when all are +1)

or –1 (when any two are –1 and one +1)

or 1 (when any two are +1 and one –1)

but –2 is not a possible value.

Illustration 18 Show that

$$\cot\frac{\pi}{20}\cot\frac{3\pi}{20}\cot\frac{5\pi}{20}\cot\frac{7\pi}{20}\cot\frac{9\pi}{20}=1$$

Solution L.H.S. $=\cot\frac{\pi}{20}\cot\frac{3\pi}{20}\cot\frac{5\pi}{20}\cot\frac{7\pi}{20}\cot\frac{9\pi}{20}$

$$=\left(\cot\frac{\pi}{20}\cot\frac{9\pi}{20}\right)\left(\cot\frac{3\pi}{20}\cot\frac{7\pi}{20}\right)\cot\left(\frac{5\pi}{20}\right)$$

$[\because\ A+B=\pi/2 \Rightarrow \cot A\cot B=1]$

$=1=$ R.H.S.

Illustration 19 Prove that

$$\cot A+\tan(180°+A)+\tan(90°+A)+\tan(360°-A)=0$$

Solution L.H.S. $=\cot A+\tan(180°+A)$

$+\tan(90°+A)+\tan(360°-A)=0.$

$=\cot A+\tan A-\cot A-\tan A=0=$ R.H.S.

Illustration 20 If $\theta=\frac{\pi}{21}$, then show that

$$\frac{\sin 23\theta-\sin 7\theta}{\sin 2\theta+\sin 14\theta}=-1.$$

Solution $\pi=21\theta$

$\sin 23\theta=\sin(21\theta+2\theta)=\sin(\pi+2\theta)=-\sin 2\theta$

$\sin 14\theta=\sin(21\theta-7\theta)=\sin(\pi-7\theta)=\sin 7\theta$

$$\text{L.H.S.}=\frac{\sin 23\theta-\sin 7\theta}{\sin 2\theta+\sin 14\theta}=\frac{-\sin 2\theta-\sin 7\theta}{\sin 2\theta+\sin 7\theta}$$

$=-1=$ R.H.S.

Illustration 21 If $\cos\theta>0$, $\tan\theta+\sin\theta=m$ and $\tan\theta-\sin\theta=n$, then show that $m^2-n^2=4\sqrt{mn}$.

Solution $\cos\theta>0$, $m=\tan\theta+\sin\theta$ and $n=\tan\theta-\sin\theta$

$\Rightarrow\ m^2-n^2=(\tan\theta+\sin\theta)^2-(\tan\theta-\sin\theta)^2$

$$=4\tan\theta\sin\theta=4\frac{\sin\theta}{\cos\theta}\cdot\sin\theta=4\frac{\sin^2\theta}{\cos\theta}$$

$$=4\sqrt{\frac{\sin^4\theta}{\cos^2\theta}}\ (\text{since}\cos\theta>0)$$

$$=4\sqrt{\frac{\sin^2\theta}{\cos^2\theta}\left(1-\cos^2\theta\right)}=4\sqrt{\tan^2\theta-\sin^2\theta}$$

$$=4\sqrt{(\tan\theta+\sin\theta)(\tan\theta-\sin\theta)}=4\sqrt{mn}$$

i.e., $m^2-n^2=4\sqrt{mn}$.

DPP 2

Total Marks 64 **Time 60 Minute**

Question Number 1 to 16. **Marking Scheme :** +4 for correct answer 0 in all other cases.

1. If $\cos\theta=\frac{3}{5}$ and θ is not in the first quadrant then show that $\frac{5\sin\theta-3\tan\theta}{3\sec\theta-4\cot\theta}=0$.

2. If $\sec\theta+\tan\theta=\frac{1}{5}$ then find the value of $\sin\theta$ and determine the quadrant in which 'θ' lies.

3. If $\tan\alpha$ and $\tan\beta$ are two solutions of $x^2-px+q=0$, $\cot\alpha$ and $\cot\beta$ are the roots of $x^2-rx+s=0$ then the value of rs is equal to :

(A) $\frac{p}{q^2}$ (B) $\frac{q}{p^2}$ (C) $\frac{1}{pq}$ (D) pq

4. The value of

$$\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right)+\sin^2\left(\frac{4\pi}{9}\right)\text{ is}$$

(A) 2 (B) 3 (C) 4 (D) None

5. If $\sin\theta+\sin^2\theta=1$, then $\cos^2\theta+\cos^4\theta=$

(A) 0 (B) 1 (C) 2 (D) None

6. Show that

$2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1=0$

7. If $0<x<\frac{\pi}{4}$ and $\cos x+\sin x=\frac{5}{4}$, find the numerical values of $\cos x-\sin x$.

8. Eliminate 'θ' from the equations

$a\cos\theta+b\sin\theta+c=0$, $a_1\cos\theta+b_1\sin\theta+c_1=0$,

where a, b, c, a_1, b_1, c_1 are real constants and $(ab_1-a_1b)\neq 0$.

9. Which of following is correct (where $n\in N$) ?

(A) $\sin\theta=\frac{n+1}{n}$ (B) $\sin\theta=\frac{n^2+1}{n+1}$

(C) $\sec\theta=\frac{n+2}{n-1}$ (D) $\sec\theta=\frac{n}{\sqrt{n^2+1}}$

10. If $\sin^2\theta_1 + \sin^2\theta_2 + ... + \sin^2\theta_n = 0$, then find the minimum value of $\cos\theta_1 + \cos\theta_2 + ... + \cos\theta_n$.

11. If $\sin^2\theta = x^2 - 3x + 3$ is meaningful, then find the values of x.

12. Prove that $(\sin\theta + \text{cosec}\,\theta)^2 + (\cos\theta + \sec\theta)^2 \geq 9$.

13. Prove that

$$\sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°) = -1.$$

14. Prove that :

(i) $\tan 720° - \cos 270° - \sin 150° \cos 120° = \frac{1}{4}$

(ii) $\sin 780° \sin 480° + \cos 120° \sin 150° = \frac{1}{2}$

15. If $\alpha = \frac{\pi}{3}$, prove that $\cos\alpha \cos 2\alpha \cos 3\alpha \cos 4\alpha \cos 5\alpha$

$\cos 6\alpha = -\frac{1}{16}$.

16. Find the value of $\tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{5\pi}{20}\tan\frac{7\pi}{20}\tan\frac{9\pi}{20}$.

Result Analysis

1. 48 to 64 Marks : **Advance Level.**
2. 32 to 47 Marks : **Main Level.**
3. < 32 Marks : **Below Average**
   (Please go through this article again)

TRIGONOMETRIC RATIOS FOR COMPOUND ANGLES

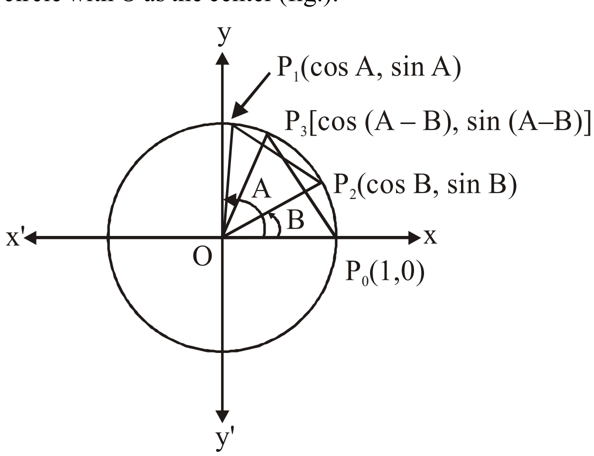
Cosine of the difference and Sum of two angles

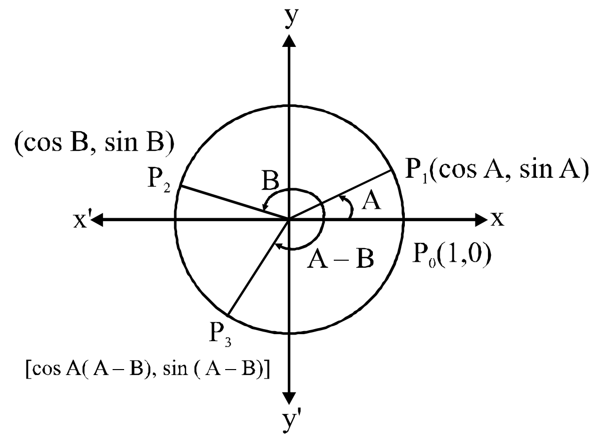
1. $\cos(A - B) = \cos A \cos B + \sin A \sin B$
2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

for all angles A and B.

Proof

1. Let X' OX and YOY' be the coordinate axes. Consider a unit circle with O as the center (fig.).





Let P_1, P_2 and P_3 be the three points on the circle such that $\angle XOP_1 = A$, $\angle XOP_2 = B$, and $\angle XOP_3 = A - B$.

As we know that the terminal side of any angle intersects the circle with center at O and unit radius at a point whose coordinates are the cosine and sine of the angle.

Therefore, coordinates of P_1, P_2, and P_3 are (cos A, sin A), (cos B, sin B), and (cos (A – B), sin (A – B)), respectively.

We know that equal chords of a circle make equal angles at its center and chords P_0P_3 and P_1P_2 subtend equal angles at O. Therefore,

Chord P_0P_3 = Chord P_1P_2

$$\Rightarrow \sqrt{\{\cos(A-B)-1\}^2 + \{\sin(A-B)-0\}^2}$$

$$= \sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$$

or $\{\cos(A-B)-1\}^2 + \sin^2(A-B)$

$= (\cos B - \cos A)^2 + (\sin B - \sin A)^2$

or $\cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B)$

$= \cos^2 B + \cos^2 A - 2\cos A \cos B + \sin^2 B + \sin^2 A$

$- 2\sin A \sin B$

or $2 - 2\cos(A-B) = 2 - 2\cos A \cos B - 2\sin A \sin B$ or $\cos(A-B) = \cos A \cos B + \sin A \sin B$

2. $\cos(A+B) = \cos(A-(-B))$ .....(i)

$= \cos A \cos(-B) + \sin A \sin(-B)$ [Using (i)]

$= \cos A \cos B - \sin A \sin B$

Hence, $\cos(A+B) = \cos A \cos B - \sin A \sin B$

[$\because$ $\cos(-B) = \cos B$, $\sin(-B) = -\sin B$]

Note : This method of proof of the above formula is true for all values of angles A and B whether the value is positive zero, or negative.

Sine of the Difference and Sum of Two Angles

1. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
2. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Proof

1. $\sin(A-B) = \cos(90° - (A-B))$ [$\because \cos(90° - \theta) = \sin\theta$]

$= \cos((90° - A) + B)$

$= \cos(90° - A)\cos B - \sin(90° - A)\sin B$

$= \sin A\cos B - \cos A\sin B$

2. $\sin(A+B) = \sin(A-(-B))$ .....(i)

$= \sin A\cos(-B) - \cos A\sin(-B)$ [Using (i)]

$= \sin A\cos B + \cos A\sin B$ [$\because \sin(-B) = -\sin B$]

Tangent of the Difference and Sum of Two Angles

1. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

2. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Proof

1. $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} =$

$\dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ [On dividing the numerator and denominator by cos A cos B]

2. $\tan(A-B) = \tan(A + (-B))$

$= \dfrac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)} = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Similarly, it can be proved that

$\cot(A+B) = \dfrac{\cot A\cos B - 1}{\cot B + \cot A}$ and

$\cot(A-B) = \dfrac{\cot A\cos B + 1}{\cot B - \cot A}$

Some more results

1. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

2. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

3. $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C$
$+ \cos A\cos B\sin C - \sin A\sin B\sin C$

4. $\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C$
$- \sin A\cos B\sin C - \sin A\sin B\cos C$

5. $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

Proof

1. $\sin(A+B)\sin(A-B) = (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B$

$= \sin^2 A - \sin^2 B$

$= (1 - \cos^2 A) - (1 - \cos^2 B) = \cos^2 B - \cos^2 A$

2. $\cos(A+B)\cos(A-B) = (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$= \cos^2 A - \sin^2 B$

$= (1 - \sin^2 A) - (1 - \cos^2 B) = \cos^2 B - \sin^2 A$

3. We have,

$\sin(A+B+C)$

$= \sin((A+B)+C)$

$= \sin(A+B)\cos C + \cos(A+B)\sin C$

$= (\sin A\cos B + \cos A\sin B)\cos C$
$+ (\cos A\cos B - \sin A\sin B)\sin C$

$= \sin A\cos B\cos C + \cos A\sin B\cos C$
$+ \cos A\cos B\sin C - \sin A\sin B\sin C$

4. We have,

$\cos(A+B+C)$

$= \cos((A+B)+C)$

$= \cos(A+B)\cos C - \sin(A+B)\sin C$

$= (\cos A\cos B - \sin A\sin B)\cos C$
$- (\sin A\cos B + \cos A\sin B)\sin C$

$= \cos A\cos B\cos C - \sin A\sin B\cos C$
$- \sin A\cos B\sin C - \cos A\sin B\sin C$

5. $\tan(A+B+C) = \tan((A+B)+C)$

$$= \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C} = \frac{\dfrac{\tan A + \tan B}{1 - \tan A\tan B} + \tan C}{1 - \left(\dfrac{\tan A + \tan B}{1 - \tan A\tan B}\right)\tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$$

Illustration 1 Show that :

(i) $\sin 15^o = \dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \cos 75^o$

(ii) $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$

(iii) $\tan 15° = 2-\sqrt{3} = \cot 75°$

(iv) $\cot 15° = 2+\sqrt{3} = \tan 75°$

Solution

(i) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$A=45°, B=30°\ \sin(45°-30°) = \sin 45°\cos 30° - \cos 45°\sin 30°$

$\Rightarrow \sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$

(ii) $\cos(45°–30°) = \cos 45°\cos 30° + \sin 45°\sin 30°$

$\cos 15° = \dfrac{\sqrt{3}}{2}, \dfrac{1}{\sqrt{2}} + \dfrac{1}{2}, \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\tan 15° = \dfrac{\sin 15°}{\cos 15°} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \dfrac{\left(\sqrt{3}-1\right)^2}{2} = 2-\sqrt{3},$

(iv) $\cot 15° = \dfrac{1}{\tan 15°} = \dfrac{1}{2-\sqrt{3}} 2+\sqrt{3}$

Illustration 2 Prove that

$$\frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} + \frac{\sin(A-B)}{\cos A\cos B} = 0.$$

Solution The first term of the L.H.S. is

$$\frac{\sin(B-C)}{\cos B\cos C} = \frac{\sin B\cos C - \cos B\sin C}{\cos B\cos C}$$

$$= \frac{\sin B\cos C}{\cos B\cos C} - \frac{\cos B\sin C}{\cos B\cos C} = \tan B - \tan C.$$

Similarly, the second term of the L.H.S. is $(\tan C - \tan A)$

and the third term of the L.H.S. is $(\tan A - \tan A)$

Now L.H.S. $= (\tan B - \tan C) + (\tan C - \tan A) + (\tan A - \tan B) = 0.$

Illustration 3 If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + \alpha = 0$, then prove that $1 + \cot\alpha\tan\beta = 0$.

Solution Given, $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$

or $\cos\alpha\cos\beta - \sin\alpha\sin\beta = 1$

or $\cos(\alpha+\beta) = 1$

Now $1 + \cot\alpha\tan\beta = 1 + \dfrac{\cos\alpha}{\sin\alpha}\times\dfrac{\sin\beta}{\cos\beta}$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta}$$

$$= \frac{\sin(\alpha-\beta)}{\sin\alpha\cos\beta}$$

$$= \frac{0}{\sin\alpha\cos\beta} = 0$$

$[\because \sin^2(\alpha+\beta) = 1 - \sin^2(\alpha+\beta) = 1 - 1 = 0]$

Illustration 4 If $\cot\beta = 2\tan(\alpha-\beta)$, then show that

$\tan\alpha = 2\tan\beta + \cot\beta$

Solution Given $\cot\beta = 2\tan(\alpha-\beta) = \dfrac{2(\tan\alpha - \tan\beta)}{1+\tan\alpha\cdot\tan\beta}$

$\Rightarrow \cot\beta(1 + \tan\alpha\tan\beta) = 2\tan\alpha - 2\tan\beta$

$\Rightarrow \cot\beta + \tan\alpha = 2\tan\alpha - 2\tan\beta$

$\Rightarrow \cot\beta + 2\tan\beta = \tan\alpha$

Illustration 5 If $A + B = \dfrac{\pi}{4}$ then show that

(i) $(1 + \tan A)(1 + \tan B) = 2$

(ii) $(\cot A - 1)(\cot B - 1) = 2$

and hence find the values of

(iii) $(1 + \tan 1°)(1 + \tan 2°)(1 + \tan 3°) \ldots (1 + \tan 44°)$

(iv) $(\cot 1° - 1)(\cot 2° - 1)(\cot 3° - 1) \ldots (\cot 44° - 1)$

(v) $\tan\dfrac{\pi}{8}$

Solution (i) Given that $A + B = \dfrac{\pi}{4}$

$\Rightarrow \tan(A+B) = 1$

$\Rightarrow \tan A + \tan B = 1 - \tan A\tan B$

$\Rightarrow 1 + \tan A + \tan B + \tan A\tan B = 1 + 1$

$\Rightarrow (1 + \tan A) + \tan(1 + \tan A) = 2$

$\Rightarrow (1 + \tan A)(1 + \tan B) = 2$

(ii) $\cot(A+B) = 1 \Rightarrow \cot B\cot A - 1 = \cot A\cot B$

$\Rightarrow \cot B\cot A - \cot A - \cot B + 1 = 1 + 1$

$\Rightarrow \cot A(\cot B - 1) - (\cot B - 1) = 2$

$\Rightarrow (\cot A - 1)(\cot B - 1) = 2$

(iii) $(1 + \tan 1°)(1 + \tan 2°) \ldots\ldots (1 + \tan 43°)(1 + \tan 44°)$

$= (1 + \tan 1°)(1 + \tan 44°)(1 + \tan 2°)(1 + \tan 43°)$

$\ldots\ldots (1 + \tan 22°)(1 + \tan 23°)$

$= 2.2.2\ldots$ (22 times) [using (ii)]

$= 2^{22}$

(iv) $(\cot 1° - 1)(\cot 2° - 1) .... (\cot 43° - 1)(\cot 44° - 1)$

$= (\cot 1° - 1)(\cot 44° - 1)(\cot 2° - 1)(\cot 43° - 1)$

$...... (\cot 22° - 1)(\cot 23° - 1)$

$= 2.2.2......2$ (22 times) [using (i)]

$= 2^{22}$

(v) Put B = A in (i) we get

$(1 + \tan A)(1 + \tan A) = 2$

$\Rightarrow 1 + \tan A = \sqrt{2}$

$\Rightarrow \tan A = \sqrt{2} - 1$

$\because A + B = \frac{\pi}{4} \Rightarrow 2A = \frac{\pi}{4} \Rightarrow A = \frac{\pi}{8}$

$\therefore \tan\frac{\pi}{8} = \sqrt{2} - 1 = \cot\frac{3\pi}{8}$

Illustration 6 Prove that : $\frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \tan 55°$.

Solution $\frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \frac{1 + \tan 10°}{1 - \tan 10°} = \frac{\tan 45° + \tan 10°}{1 - \tan 45° \tan 10°}$

$= \tan(45° + 10°) = \tan 55°$ (dividing by cos 10°)

Illustration 7 Prove that $\tan 70° = 2\tan 50° + \tan 20°$

Solution $\tan 70° = \tan(50° + 20°) = \frac{\tan 50° + \tan 20°}{1 - \tan 50° \tan 20°}$

or $\tan 70°(1 - \tan 50° \tan 20°) = \tan 50° + \tan 20°$

or $\tan 70° - \tan 50° \tan 20° \tan 70° = \tan 50° + \tan 20°$

or $\tan 70° = \tan 70° \tan 50° \tan 20° + \tan 50° + \tan 20°$

$= \tan(90°–20°) \tan 50° \tan 20° + \tan 50° + \tan 20°$

$= \cot 20° \tan 50° \tan 20° + \tan 50° + \tan 20°$

$= \tan 50° + \tan 50° + \tan 20° = 2\tan 50° + \tan 20°$

Illustration 8 If $\sin(A - B) = \frac{1}{\sqrt{10}}, \cos(A + B) = \frac{2}{\sqrt{29}}$, find the value of tan 2A where A and B lie between 0 and $\pi/4$.

Solution $\tan 2A = \tan[(A + B) + (A - B)]$

$= \frac{\tan(A + B) + \tan(A - B)}{1 - \tan(A + B)\tan(A - B)}$

Given that, $0 < A < \pi/4$, and $0 < B < \pi/4$. There,

$0 < A + B < \frac{\pi}{2}$

Also, $-\frac{\pi}{4} < A - B < \frac{\pi}{4}$ and $\sin(A - B) = \frac{1}{\sqrt{10}} = (+)$ ve

$\therefore 0 < A - B < \frac{\pi}{4}$

Now, $\sin(A - B) = \frac{1}{\sqrt{10}}$

$\Rightarrow \tan(A - B) = \frac{1}{3}$

$\Rightarrow \cos(A + B) = \frac{2}{\sqrt{29}}$

$\Rightarrow \tan(A + B) = \frac{5}{2}$

From Eq.. (i), (ii) and (iii), we get

$$\tan 2A = \frac{\frac{5}{2} + \frac{1}{3}}{1 - \frac{5}{2} \times \frac{1}{3}} = \frac{17}{6} \times \frac{6}{1} = 17$$

Illustration 9 Prove that : $\frac{\tan^2 2\theta - \tan^2\theta}{1 - \tan^2 2\theta \tan^2\theta} = \tan 3\theta \tan\theta$

Solution $\frac{\tan^2 2\theta - \tan^2\theta}{1 - \tan^2 2\theta \tan^2\theta}$

$= \left(\frac{\tan 2\theta - \tan\theta}{1 + \tan 2\theta \tan\theta}\right)\left(\frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}\right)$

$= (\tan 2\theta - \theta)\tan(2\theta + \theta) = \tan 3\theta \tan\theta$

Illustration 10 Find the value of

$$\cos\frac{\pi}{12}\left(\sin\frac{5\pi}{12} + \cos\frac{\pi}{4}\right) + \sin\frac{\pi}{12}\left(\cos\frac{5\pi}{12} - \sin\frac{\pi}{4}\right).$$

Solution $\cos 15°(\sin 75° + \cos 45°) + \sin 15°(\cos 75° - \sin 45°)$

$= \sin(75° + 15°) + \cos(45° + 15°) = 1 + \frac{1}{2} = \frac{3}{2}$

DPP 3

Total Marks 50 **Time 60 Minute**

Question Number 1 to 9. **Marking Scheme :** +4 for correct answer 0 in all other cases.

1. If $A + B = \frac{3\pi}{4}$ then prove that **[10 Marks]**

(i) $(\tan A - 1)(\tan B - 1) = 2$

(ii) $(\cot A + 1)(\cot B + 1) = 2$

Hence, deduce the value of

(iii) $(\tan 46° - 1)(\tan 47° - 1)(\tan 48° - 1)....(\tan 89° - 1)$

(iv) pq where p = (cot 1° + 1)

(cot 2° + 1) ...(cot 9° +1)

q = (cot 134° + 1)

(cot 133° + 1)...(cot 126° + 1)

(v) $\cot 67\frac{1}{2}^{\circ}$

2. Prove that **[12 Marks]**

(i) tan A + tan B + tan A tan B tan (A + B) = tan (A + B)

(ii) tan A + 2 tan B

$= \tan(A + B)$ or $\cot A$ If $2A + B = \frac{\pi}{2}$

Hence find the values of

(iii) $\tan 23° + \tan 37° + \sqrt{3}\tan 23° \tan 37°$.

(iv) tan 20° + 2 tan 50°.

(v) 2 tan 40° + tan 20° + 4 tan 10°

(vi) tan 18° + tan 27° + tan 18 tan 7°

3. Show that $\cos^2\theta + \cos^2(\alpha + \theta) - 2\cos\alpha\cos\alpha\cos\theta\cos(\alpha + \theta)$ is independent of θ.

4. Let A, B, C be the three angles such that A + B + C = π.

If tan A · tan B = 2. Then find the value of $\frac{\cos A \cos B}{\cos C}$.

5. If sinA + cos 2A = 1/2 and cos A + sin 2A = 1/3, then find the value of sin 3A.

6. If sin x + sin y + sin z = 0 = cos x + cos y + cos z, then find the value of expression cos θ.

7. If cos(α + β) + sin (α – β) = 0 and tan β ≠ 1, then find the value of tan α.

8. If x is A.M. of tan π/9 and tan 5π/18 and y is A.M. of tan π/9 and 7π/18, then relate x and y.

9. If tan A = 1/2, tan B = 1/3, then prove that cos 2A = sin 2B.

Result Analysis

1. 32 to 50 Marks : **Advance Level.**
2. 22 to 31 Marks : **Main Level.**
3. < 22 Marks : **Below Average**
   (Please go through this article again)

TRANSFORMATION FORMULA

Formula to Transform the Product into Sum or Difference

We know that

$$\sin A\cos B + \cos A\sin B = \sin(A + B) \quad .....(i)$$

$$\sin A\cos B - \cos A\sin B = \sin(A - B) \quad .....(ii)$$

$$\cos A\cos B - \sin A\sin B = \cos(A + B) \quad .....(iii)$$

$$\cos A\cos B + \sin A\sin B = \cos(A - B) \quad .....(iv)$$

Adding Eqs. (i) and (ii), we obtain

$$2\sin A\cos B = \sin(A + B) + \sin(A - B) \quad .....(v)$$

Substracting Eqs.; (ii) from (i), we get

$$2\cos A\sin B = \sin(A + B) - \sin(A - B) \quad .....(vi)$$

Adding Eqs. (iii) and (iv), we get

$$2\cos A\cos B = \cos(A + B) + \cos(A - B) \quad .....(vii)$$

Subtracting Eqs. (iii) from (iv), we get

$$2\sin A\sin B = \cos(A - B) - \cos(A + B) \quad .....(viii)$$

Formulas to Transform the Sum of Difference into Product

Let A + B = C and A – B = D.

Then, $A = \frac{C + D}{2}$ and $B = \frac{C - D}{2}$

Substituting the values of A, B, C and D in Eqs. (v), (vi), (vii) and (viii), we get

$$\sin C + \sin D = 2\sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)$$

$$\sin C - \sin D = 2\sin\left(\frac{C - D}{2}\right)\cos\left(\frac{C + D}{2}\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right) \quad .....(xi)$$

$$\left.\begin{aligned} \cos D - \cos C &= 2\sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right) \\ \cos C - \cos D &= -2\sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right) \\ \cos C - \cos D &= 2\sin\left(\frac{C + D}{2}\right)\sin\left(\frac{D - C}{2}\right) \end{aligned}\right\} \quad .....(xii)$$

These four formulas are used to convert the sum or difference of two sines or two cosines into the product of sines and cosines.

Illustration 1 Prove that :

(i) $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$ (ii) $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$

Solution

(i) L.H.S.

$$= \frac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \frac{2\sin\left(\frac{5A-3A}{2}\right)\cos\left(\frac{5A+3A}{2}\right)}{2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}$$

$$= \frac{2\sin A\cos 4A}{2\cos 4A\cos A} = \tan A = \text{R.H.S.}$$

(ii) L.H.S. $= \dfrac{\sin 3A + \sin A}{\cos 3A + \cos A}$

$$= \frac{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)}{2\cos\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)}$$

$$= \frac{\sin 2A\cos A}{\cos 2A\cos A} = \tan 2A = \text{R.H.S.}$$

Illustration 2 Prove that $\cos 18° - \sin 18° = \sqrt{2}\sin 27^o$

Solution L.H.S. $= \cos 18° - \sin 18°$

$= \cos 18° - \sin(90° - 72°) = \cos 18° - \cos 72°$

$$= 2\sin\frac{18^o + 72^o}{2}\sin\frac{72^o - 18^o}{2} = 2\sin 45°\sin 27°$$

$$= 2\frac{1}{\sqrt{2}}\sin 27° = \sqrt{2}\sin 27°$$

Illustration 3 (i) Prove that $\sin 65° + \cos 65° = \sqrt{2}\cos 20^o$

(ii) Prove that $\sin 47° + \cos 77° = \cos 17°$

Solution (i) $\sin 65° + \cos 65° = \sin 65° + \sin 25°$

$$= 2\sin\left(\frac{65^o + 25^o}{2}\right)\cos\left(\frac{65^o - 25^o}{2}\right)$$

$$= 2\sin 45°\cos 20° = 2\frac{1}{\sqrt{2}}\cos 20^o = \sqrt{2}\cos 20^o$$

(ii) $\sin 47° + \cos 77° = \sin 47° + \sin 13°$

$$= 2\sin\left(\frac{47^o + 13^o}{2}\right)\cos\left(\frac{47^o - 13^o}{2}\right)$$

$= 2\sin 30°\cos 17°$

$$= 2\left(\frac{1}{2}\right)\cos 17^o = \cos 17^o$$

Illustration 4 Prove that : $\cos 80° + \cos 40° - \cos 20° = 0$

Solution $\cos 80° + \cos 40° - \cos 20°$

$$= 2\cos\left(\frac{80^o + 40^o}{2}\right)\cos\left(\frac{80^o - 40^o}{2}\right) - \cos 20^o$$

$$= 2\cos 60^o\cos 20^o - \cos 20^o$$

$$= 2\frac{1}{2}\cos 20^o - \cos 20^o = 0$$

Illustration 5 Prove that

$\sin 10° + \sin 20° + \sin 40° + \sin 50° = \sin 70° + \sin 80°$

Solution $\sin 10° + \sin 20° + \sin 40° + \sin 50°$

$$= 2\sin\left(\frac{10^o + 50^o}{2}\right)\cos\left(\frac{50^o - 10^o}{2}\right)$$

$$+ 2\sin\left(\frac{20^o + 40^o}{2}\right)\cos\left(\frac{40^o - 20^o}{2}\right)$$

$= 2\sin 30°\cos 20° + 2\sin 30°\cos 10°$

$$= 2\frac{1}{2}\sin 70^o + 2\frac{1}{2}\sin 80^o$$

$$= \sin 70^o + \sin 80^o$$

Illustration 6 Prove that :

$$\cos\frac{\pi}{5} + \cos\frac{2\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{7\pi}{5} = 0$$

Solution $\cos\frac{\pi}{5} + \cos\frac{2\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{7\pi}{5} = 0$

$$= \left(\cos\frac{\pi}{5} + \cos\frac{7\pi}{5}\right) + \left(\cos\frac{2\pi}{5} + \cos\frac{6\pi}{5}\right)$$

$$= 2\cos\frac{4\pi}{5}\cos\frac{3\pi}{5} + 2\cos\frac{4\pi}{5}\cos\frac{2\pi}{5}$$

$$= 2\cos\frac{4\pi}{5}\left(\cos\frac{3\pi}{5} + \cos\frac{2\pi}{5}\right)$$

$$= 2\cos\frac{4\pi}{5}\left(2\cos\frac{\pi}{2} + \cos\frac{\pi}{10}\right) = 0$$

Illustration 7 Prove that :

$$\frac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A} = \tan 3A.$$

Solution $\dfrac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A}$

$$=\frac{(\sin 5A+\sin A)+(\sin 4A+\sin 2A)}{(\cos 5A+\cos A)+(\cos 4A+\cos 2A)}$$

$$=\frac{2\sin 3A\cos 2A+2\sin 3A\cos A}{2\cos 3A\cos 2A+2\cos 3A\cos A}$$

$$=\frac{2\sin 3A(\cos 2A+\cos A)}{2\cos 3A(\cos 2A+\cos A)}=\tan 3A$$

Illustration 8 Prove that

$$(\cos\alpha+\cos\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right) + (\sqrt{\sin\alpha+\sin\beta})^2.$$

Solution L.H.S. = $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2$

$$=\left\{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2+\left\{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2$$

$$=4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left\{\cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}\right\}$$

$$=4\cos^2\left(\frac{\alpha-\beta}{2}\right) \qquad \left[\because \cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}=1\right]$$

= R.H.S.

Illustration 9 In quadrilateral ABCD if

$$\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)=2,$$

then find the value of $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}$.

Solution

$$\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)=2$$

or $\frac{1}{2}[\sin A+\sin B+\sin C+\sin D]=2$

or $\sin A+\sin B+\sin C+\sin D=4$

or $A=B=C=D=90^\circ$

or $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}=\frac{1}{4}$

Illustration 10 Prove that $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$$=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}.$$

Solution L.H.S. = $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$$=(\cos\alpha+\cos\beta)+[\cos\gamma+\cos(\alpha+\beta+\gamma)]$$

$$=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$+2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$$

$$=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)+2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)$$

$$=2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$$

$$=2\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\left\{2\cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2}-\frac{\alpha-\beta}{2}}{2}\right)\right\}$$

$$=2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\}$$

$$=4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right)=\text{R.H.S.}$$

Illustration 11 Prove that :

$$\left(\frac{\cos A-\cos B}{\sin A+\sin B}\right)^n-\left(\frac{\sin A-\sin B}{\cos A+\cos B}\right)^n$$

$$=\begin{cases}0 & \text{if } n \text{ is even}\\ -2\tan^n\left(\frac{A-B}{2}\right) & \text{if } n \text{ is odd}\end{cases}$$

Solution $\left(\dfrac{\cos A-\cos B}{\sin A+\sin B}\right)^n-\left(\dfrac{\sin A-\sin B}{\cos A+\cos B}\right)^n$

$$=\left[\frac{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}\right]^n-$$

$$\left[\frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}\right]^n$$

$$=(-1)^n\tan^n\left(\frac{A-B}{2}\right)-\tan^n\left(\frac{A-B}{2}\right)$$

$$=\begin{cases}0 \text{ if n is even, since } (-1)^n=1\\ -2\tan^n\left(\frac{A-B}{2}\right) \text{ if n is odd, since } (-1)^n=-1\end{cases}$$

Illustration 12 Suppose that $\alpha - \beta$ is not an odd multiple of $\frac{\pi}{2}, m \in R - \{0, -1\}$ and $\frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{1-m}{1+m}$.

The show that $\tan\left(\frac{\pi}{4} - \alpha\right) = m\tan\left(\frac{\pi}{4} + \beta\right)$.

Solution Since $\frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{1-m}{1+m}$ and $(1-m)-(1+m) = -2m \neq 0$,

by componendo and dividendo, we have

$$\frac{\sin(\alpha+\beta)+\cos(\alpha-\beta)}{\sin(\alpha+\beta)-\cos(\alpha-\beta)} = \frac{1-m+1+m}{1-m-1-m}$$

$$\therefore \frac{\cos\left[\frac{\pi}{2} - (\alpha+\beta)\right] + \cos(\alpha-\beta)}{\cos\left[\frac{\pi}{2} - (\alpha+\beta)\right] - \cos(\alpha-\beta)} = \frac{2}{-2m}$$

$$\Rightarrow \frac{2\cos\left(\frac{\frac{\pi}{2}-\alpha-\beta+\alpha-\beta}{2}\right)\cos\left(\frac{\frac{\pi}{2}-\alpha-\beta-\alpha+\beta}{2}\right)}{-2\sin\left(\frac{\frac{\pi}{2}-\alpha-\beta+\alpha-\beta}{2}\right)\sin\left(\frac{\frac{\pi}{2}-\alpha-\beta-\alpha+\beta}{2}\right)}$$

$$= -\frac{1}{m}$$

$$\therefore \frac{\cos\left(\frac{\pi}{4}-\beta\right)\cos\left(\frac{\pi}{4}-\alpha\right)}{\sin\left(\frac{\pi}{4}-\beta\right)\sin\left(\frac{\pi}{4}-\alpha\right)} = \frac{1}{m}$$

$$\therefore \frac{\cot\left(\frac{\pi}{4}-\beta\right)}{\tan\left(\frac{\pi}{4}-\alpha\right)} = \frac{1}{m}$$

$$\therefore \tan\left(\frac{\pi}{4}-\alpha\right) = m\cot\left(\frac{\pi}{4}-\beta\right) = m\tan\left(\frac{\pi}{4}+\beta\right)$$

DPP 4

Total Marks 32 **Time 30 Minute**

Question Number 1 to 8. **Marking Scheme :** +4 for correct answer 0 in all other cases.

1. If sin A = sin B and cos A = cos B, then prove that $\sin\frac{A-B}{2} = 0$.
2. If $\sec(\theta+\alpha) + \sec(\theta-\alpha) = 2\sec\theta$, then show that $\cos^2\theta = 1 + \cos\alpha$.
3. If sin α – sin β = 1/3 an cos β – cos α = 1/2, show that $\cot\frac{\alpha+\beta}{2} = \frac{2}{3}$
4. If cosec A + sec A = cosec B + sec B,

   Prove that tan A tan B = $\cot\frac{A+B}{2}$
5. Prove that sin 25° cos 115° = 1/2 (sin 40° – 1).
6. If cos A = 3/4, then find the value of $32\sin\left(\frac{A}{2}\right)\sin\left(\frac{5A}{2}\right)$.
7. If cos (A + B) sin (C + D) = cos (A – B) sin (C – D), prove that cot A cot B cot C = cot D.
8. If tan(A + B) = 3 tan A, prove that
   (i) sin(2A + B) = 2 sin B
   (ii) sin2(A + B) + sin 2A = 2 sin 2B

Result Analysis

1. 24 to 32 Marks : **Advance Level.**
2. 16 to 23 Marks : **Main Level.**
3. < 16 Marks : **Below Average**
   (Please go through this article again)

TRIGONOMETRIC RATIOS OF MULTIPLES AND SUB-MULTIPLE ANGLES

Formulas for Multiple Angles

1. $\cos 2A = \cos(A + A) = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A = 2\cos^2 A - 1$

Also $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$,

$\cos^2 A = \frac{1}{2}(1+\cos 2A)$

2. $\sin 2A = \sin(A + A)$

$= \sin A \cos A + \sin A \cos A$

$= 2\sin A \cos A$

3. $\tan 2A = \tan(A + A)$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2\tan A}{1-\tan^2 A}$$

4. $\sin 3A = \sin(2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= 2\sin A \cos A \cos A + (1-2\sin^2 A)\sin A$

$= 2\sin A \cos^2 A + \sin A - 2\sin^3 A$

$= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$

$= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A$

5. $\cos 3A = \cos(2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin A$

$\cos A \sin A$

$= 2\cos^3 A - \cos A - 2\cos A$

$(1 - \cos^2 A)$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

6. $\sin 2A$ and $\cos 2A$ in terms of $\tan A$

$\sin 2A = 2\sin A \cos A$

$$= \frac{2\sin A\cos A}{\cos^2 A + \sin^2 A} = \frac{2\tan A}{1+\tan^2 A}$$

[Dividing numerator and denominator by $\cos^2 A$]

$\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1-\tan^2 A}{1+\tan^2 A}$$

[Dividing numerator and denominator by $\cos^2 A$]

Also $\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

7. In the formula of $\tan(A + B + C)$, putting $B = A$ and $C = A$,

We get $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Similarly, we can prove that $\cot 3A = \frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

8. $\tan(A_1 + A_2 + ... + A_n) = \frac{S_1 - S_3 + S_5 - S_7 + ...}{1 - S_2 + S_4 - S_6 + ...}$,

where

$S_1 = \tan A_1 + \tan A_2 + ... + \tan A_n$ = Sum of the tangents of the separate angles,

$S_2 = \tan A_1 \tan A_2 + \tan A_1 + \tan A_3 + ...$

= Sum of the product of tangents taken two at a time,

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + ...$

= Sum of the product of tangents taken three at a time, and so on.

If $A_1 = A_2 = ... = A_n = A$, then we have

$S_1 = n\tan A$, $S_2 = {}^nC_2 \tan^2 A$, $S_3 = {}^nC_3 \tan^3 A$,.....

Illustration 1 Prove that

(i) $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

(ii) $\frac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$

(iii) $\frac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \cot\theta$

(iv) $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\theta/2$

(v) $\frac{\cos 2\theta}{1+\sin 2\theta} = \tan(\pi/4-\theta)$

(vi) $\frac{\cos\theta}{1+\sin\theta} = \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

Solution (i) L.H.S. $= \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \tan\theta$ = R.H.S.

(ii) L.H.S. $= \frac{\sin 2\theta}{1-\cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\sin^2\theta} = \cot\theta$ = R.H.S.

(iii) L.H.S. $= \frac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \frac{(1+\cos 2\theta)+\sin 2\theta}{(1-\cos 2\theta)+\sin 2\theta}$

$$= \frac{2\cos^2\theta + 2\sin\theta\cos\theta}{2\sin^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{2\cos\theta(\cos\theta+\sin\theta)}{2\sin\theta(\cos\theta+\sin\theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{R.H.S.}$$

(iv) $\text{L.H.S.} = \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \frac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$

$$= \frac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}$$

$$= \tan\frac{\theta}{2} = \text{R.H.S.}$$

(v) $\text{L.H.S.} = \frac{\cos 2\theta}{1+\sin 2\theta} = \frac{\sin\left(\frac{\pi}{2}-2\theta\right)}{1+\cos\left(\frac{\pi}{2}-2\theta\right)}$

$$= \frac{2\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{2\cos^2\left(\frac{\pi}{4}-\theta\right)} = \tan\left(\frac{\pi}{4}-\theta\right) = \text{R.H.S.}$$

(vi) L.H.S.

$$= \frac{\cos\theta}{1+\sin\theta} = \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \text{R.H.S.}$$

Illustration 2 Show that $\sqrt{3}$ cosec 20º – sec20º = 4.

Solution $\text{L.H.S.} = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ} = \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{\sin 20^\circ\cos 20^\circ}$$

$$= \frac{2(\sin 60^\circ\cos 20^\circ - \cos 60^\circ \sin 20^\circ)}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\sin(60^\circ - 20^\circ)}{\sin 20^\circ\cos 20^\circ} = \frac{2\sin 40^\circ}{\sin 20^\circ\cos 20^\circ}$$

$$= \frac{4\sin 40^\circ}{2\sin 20^\circ\cos 20^\circ} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4$$

Illustration 3 Expand cos 5θ terms of θ.

Solution $\cos 5\theta = \cos(3\theta + 2\theta)$

$= \cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta$

$= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1)$
$\quad - (3\sin\theta - 4\sin^3\theta)\cdot 2\sin\theta\cos\theta$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta)$
$\quad - 2\cos\theta\cdot\sin^2\theta(3 - 4\sin^2\theta)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta)$
$\quad - 2\cos\theta\cdot\sin^2\theta(3 - 4\sin^2\theta)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta)$
$\quad - 2\cos\theta(1 - \cos^2\theta)(4\cos^2\theta - 1)$

$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta)$
$\quad - 2\cos\theta(5\cos^2\theta - 4\cos^4\theta - 1)$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Illustration 4 Find the values of

(i) $\sin 22\frac{1^\circ}{2}$ (ii) $\cos 22\frac{1^\circ}{2}$

(iii) $\tan 22\frac{1^\circ}{2}$ (iv) $\cot 22\frac{1^\circ}{2}$

Solution (i) $\sin 22\frac{1^\circ}{2} = \sqrt{\frac{1-\cos 45^\circ}{2}}$

$$= \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \text{ (since } \sin 22\frac{1^\circ}{2}\text{)}$$

(ii) Similarly we can show that

$$\cos 22\frac{1^\circ}{2} = \sqrt{\frac{\cos 2\theta + 1}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

(iii) $\tan 22\frac{1^\circ}{2} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = (\sqrt{2}-1) \left(\because \frac{\pi}{8} \in Q_1\right)$

(iv) $\cot 22\frac{1^\circ}{2} = \sqrt{2}+1 = \left(\text{since } \cot 22\frac{1^\circ}{2} > 0\right)$

Illustration 5 Show that

$$\cos^2\alpha + \cos^2(\alpha+120^\circ) + \cos^2(\alpha-120^\circ) = \frac{3}{2}$$

Solution L.H.S.

$$= \frac{1+\cos 2\alpha}{2} + \frac{1+\cos(2\alpha+240^\circ)}{2} + \frac{1+\cos(2\alpha-240^\circ)}{2}$$

$$= \frac{3}{2} + \frac{1}{2}[\cos 2\alpha + \cos(2\alpha+240^\circ) + \cos(2\alpha-240^\circ)]$$

$$= \frac{3}{2} + \frac{1}{2}[\cos 2\alpha + 2\cos 2\alpha \ \cos 240^\circ]$$

$$= \frac{3}{2} + \frac{1}{2}\left[\cos 2\alpha + 2\cos 2\alpha\left(-\frac{1}{2}\right)\right] = \frac{3}{2} = \text{R.H.S.}$$

Important result

(i) $\sin\theta \sin(60^\circ-\theta)\sin(60^\circ+\theta) = \frac{1}{4}\sin 3\theta$

(ii) $\cos\theta \cos(60^\circ-\theta)\cos(60^\circ+\theta) = \frac{1}{4}\cos 3\theta$

(iii) $\tan\theta \tan(60^\circ-\theta)\tan(60^\circ+\theta) = \tan 3\theta$

Proof (i) $\sin\theta \sin(60^\circ-\theta)\sin(60^\circ+\theta)$

$= \sin\theta[\sin^2 60^\circ - \sin^2\theta]$

($\because \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$)

$$= \sin\theta\left[\frac{3}{4} - \sin^2\theta\right] = \frac{1}{4}\left[3\sin\theta - 4\sin^3\theta\right] = \frac{1}{4}\sin 3\theta$$

(ii) $\cos\theta\cos(60^\circ-\theta)\cos(60^\circ+\theta) = \cos\theta[\cos^2 60^\circ - \sin^2\theta]$

$$= \cos\theta\left[\frac{1}{4} - 1(1-\cos^2\theta)\right]$$

($\because \cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$)

$$\cos\theta\left[-\frac{3}{4} + \cos^2\theta\right] = \frac{1}{4}\left[4\cos^3\theta - 3\cos\theta\right] = \frac{1}{4}\cos 3\theta$$

(iii) $\tan\theta\tan(60^\circ-\theta)\tan(60^\circ+\theta)$

$$= \frac{\sin\theta\ \sin(60^\circ-\theta)\sin(60^\circ+\theta)}{\cos\theta\ \cos(60^\circ-\theta)\cos(60^\circ+\theta)} = \frac{\frac{1}{4}\sin 3\theta}{\frac{1}{4}\cos 3\theta} = \tan 3\theta$$

Illustration 6 Show that :

Hence find the values of the following.

(i) $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ$

(ii) $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ$

(iii) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

(iv) $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ$

Solution (i) Put $\theta = 6^\circ, 18^\circ$ in (ii), we get

$$\sin 6^\circ \sin 54^\circ \sin 66^\circ = \frac{1}{4}\sin 36^\circ = \frac{1}{4}\sin 18^\circ \quad .....(i)$$

$$\sin 18^\circ \sin 42^\circ \sin 78^\circ = \frac{1}{4}\sin 3.18^\circ = \frac{1}{4}\sin 54^\circ \quad .....(ii)$$

(i) × (ii) $\Rightarrow \sin 6^\circ \sin 18^\circ \sin 42^\circ \sin 54^\circ \sin 66^\circ \sin 78^\circ$

$$= \frac{1}{16}\sin 18^\circ \sin 54^\circ$$

$$\Rightarrow \sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = \frac{1}{16}$$

or $\cos 84^\circ \cos 48^\circ \cos 24^\circ \cos 12^\circ = \frac{1}{16}$

(ii) Put $\theta = 6^\circ, 18^\circ$ in (iii), we get

$\tan 6^\circ \tan 54^\circ \tan 66^\circ = \tan 18^\circ$ .....(i)

$\tan 18^\circ \tan 42^\circ \tan 78^\circ = \tan 54^\circ$ .....(ii)

(i) × (ii) $\Rightarrow \tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1$

(iii) Put $x = 20^\circ$ in (i),

We get $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{1}{4}\sin 60^\circ = \frac{\sqrt{3}}{8}$

$$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$$

(iv) Put $x = 10^\circ$ in (ii),

We get $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{3}{16}$

Illustration 7 Show that

(i) $\sin^3\theta + \sin^3(60^\circ-\theta) + \sin^3(240^\circ+\theta) = -\frac{3}{4}\sin 3\theta$

(ii) $\cos^3\theta + \cos^3(120^\circ+\theta) + \cos^3(240^\circ+\theta) = \frac{3}{4}\cos 3\theta$

Hence, find the values of the following.

(iii) $\sin^3 10^\circ + \sin^3 50^\circ - \sin^3 70^\circ$

(iv) $\cos^2 20^\circ - \cos^3 40^\circ - \cos^3 80^\circ$

(v) $\dfrac{\sin^3 6^\circ + \sin^3 54^\circ - \sin^3 66^\circ}{\cos^3 12^\circ - \cos^3 48^\circ - \cos^3 72^\circ}$

Solution (i) Let $a = \sin\theta$, $b = \sin(60° - \theta)$,

$c = \sin(240° + \theta)$ then $c = -\sin(60° + \theta)$

Now $a + b + c = \sin\theta + \sin 60° - \theta) - \sin(60° + \theta)$

$= \sin\theta - 2\cos 60°\sin\theta = 0$

$\Rightarrow$ $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

$\Rightarrow$ $\sin^3\theta + \sin^3(60° - \theta) + \sin^3(240° + \theta)$

$= 3\cdot\sin\theta\sin(60° - \theta)[-\sin(60° + \theta)]$

$= -\frac{3}{4}\sin 3\theta.$

(ii) Let $a = \cos\theta$, $b = \cos(120° + \theta)$,

$c = \cos(240° + \theta)$ then

$b = \cos(180° - (60° - \theta)) = -\cos(60° - \theta)$

$c = \cos(180° + 60° + \theta) = -\cos(60° + \theta)$

$a + b + c = \cos\theta - \cos(60° - \theta) - \cos(60° + \theta)$

$= \cos\theta - 2\cos 60°\cos\theta = 0$

$\Rightarrow$ $a^3 + b^3 + c^3 = 3abc$

$\Rightarrow$ $\cos^3\theta + \cos^3(120° + \theta) + \cos^3(240° + \theta)$

$= 3\cos\theta\cos(120° + \theta)\cos(240° + \theta)$

$= 3\cos\theta\cos(60° + \theta)\cos(60° - \theta) = \frac{3}{4}\cos 3\theta$

(iii) Put $\theta = 10°$ in (i), we get

$\sin^3 10° + \sin^3 50° + \sin^3 250° = -\frac{3}{4}\sin 30° = -\frac{3}{8}$

or $\sin^3 10° + \sin^3 50° - \sin^3 70° = -\frac{3}{8}$

(iv) Put $\theta = 20°$ in (ii) we get

$\cos^3 20° + \cos^3 140° + \cos^3 260° = \frac{3}{4}\cos 60° = \frac{3}{8}$

$\cos^3 20° - \cos^3 40° + \cos^3 80° = \frac{3}{8}$

(v) $\frac{\sin^3 6° + \sin^3 54° - \sin^3 66°}{\cos^3 12° - \cos^3 48° - \cos^3 72°} = \frac{-\frac{3}{8}\sin 18°}{\frac{3}{8}\cos 36°}$

$$= -\left[\frac{\sqrt{5}-1}{\sqrt{5}+1}\right] = -\left[\frac{\left(\sqrt{5}-1\right)^2}{4}\right] = -\left[\frac{5+1-2\sqrt{5}}{4}\right]$$

$$= -\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\sqrt{5}-\sqrt{9}}{2}$$

Illustration 8 Prove that :

(i) $\tan x + \tan(60° + x) - \tan(60° - x) = 3\tan 3x$

(ii) $\cot x + \cot(60° + x) - \cot(60° - x) = 3\cot 3x$

Hence , find the values of the following.

(iii) $(\tan 10° + \tan 70° - \tan 50°)^2$

(iv) $\cot 10° + \cot 70° - \cot 50°$

Solution (i) First, $\tan(60° + x) - \tan(60° - x)$

$$= \frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x} - \frac{\sqrt{3} - \tan x}{1 + \sqrt{3}\tan x}$$

$$= \frac{\left(\sqrt{3} + 3\tan x + \tan x + \sqrt{3}\tan^2 x - \sqrt{3} + 3\tan x + \tan x - \sqrt{3}\tan^2 x\right)}{1 - 3\tan^2 x}$$

$$= \frac{8\tan x}{1 - 3\tan^2 x}$$

Therefore, LHS $= \tan x + \frac{8\tan x}{1 - 3\tan^2 x}$

$$= \frac{\tan x - 3\tan^3 x + 8\tan x}{1 - 3\tan^2 x}$$

$$= 3\left[\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right] = 3\tan 3x = \text{RHS}$$

(ii) Replace x by $30° - x$ in (i), we get

$\tan(30° - x) + \tan(90° - x) - \tan(90° - 3x)$

$\Rightarrow$ $\cot(90° - (30° - x) + \cot x$

$- \cot(90° - (30° + x)) = 3\cot 3x$

$\Rightarrow$ $\cot(60° + x) + \cot x - \cot(60° - x) = 3\cot 3x$

(iii) Put $x = 10°$ in (i), we get

$\tan 10° + \tan 70° - \tan 50° = 3\tan 30°$

$\Rightarrow$ $(\tan 10° + \tan 70° - \tan 50°)^2 = 3$

(iv) put $x = 10°$ in (iii)

We get $\cot 10° + \cot 70° - \cot 50° = 3\cot 30° = 3\sqrt{3}$

(i) $\tan x + \tan(60° + x) + \tan(120° + x) = 3\tan 3x$

(ii) $\cot x + \cot(60° + x) + \cot(120° + x = 3\cot 3x$

Illustration 9 Show that :

(i) $\sqrt{2 + 2\cos\theta} = 2\cos\frac{\theta}{2}$ where $\frac{\theta}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\sqrt{2+\sqrt{2+2\cos\theta}} = 2\cos\frac{\theta}{2^2}$ where $\frac{\theta}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) $\sqrt{2+\sqrt{2+\sqrt{2+2\cos\theta}}} = 2\cos\frac{\theta}{2^3}$ where $\frac{\theta}{2^2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

and hence deduce that

(iv) $\sqrt{2+\sqrt{2+\sqrt{2+....+\sqrt{2+2\cos\theta}}}}$ (n – radical involved

$= 2\cos\frac{\theta}{2^n}$ where $\frac{\theta}{2^{n-1}} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Solution (i) $\sqrt{2+2\cos\theta} = \sqrt{2(1+\cos\theta)} = \sqrt{2.2\cos^2\frac{\theta}{2}}$

$= 2\left|\cos\frac{\theta}{2}\right| = 2\cos\frac{\theta}{2}$ for $\frac{\theta}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\sqrt{2+\sqrt{2+2\cos\theta}} = \sqrt{2+2\cos\frac{\theta}{2}} = \sqrt{2\left(1+\cos\frac{\theta}{2}\right)}$

$= \sqrt{4\cos^2\frac{\theta}{2^2}} = 2\cos\frac{\theta}{2^2}$ for $\frac{\theta}{2^2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) $\sqrt{2+\sqrt{2+\sqrt{2+2\cos\theta}}} = \sqrt{2+2\cos\frac{\theta}{2^2}} = 2\cos\frac{\theta}{2^3}$

For $\frac{\theta}{2^2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Thus, we can conclude that

$\sqrt{2+\sqrt{2+\sqrt{2+...+2+2\cos\theta}}} = 2\cos\frac{\theta}{2^n}$

for $\frac{\theta}{2^{n-1}} \cdot \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2+2\cos 2^n\theta}}}}$

$= 2\cos\theta$ for $\theta \in \left[-\frac{\pi}{2^n}, \frac{\pi}{2^n}\right]$

Illustration 10 $\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}$ (n – radicals involved)

(A) $2\cos\frac{\pi}{2^n}$ (B) $2\cos\frac{\pi}{2^{n+1}}$

(C) $\cos\frac{\pi}{2^{n+1}}$ (D) $2\sin\frac{\pi}{2^{n+1}}$

Solution (C)

Put $\theta = \frac{\pi}{2}$ in the previous illustration,

We get $\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}} = 2\cos\left(\frac{\pi}{2}\frac{1}{2^n}\right) = 2\cos\frac{\pi}{2^{n+1}}$

Illustration 11 Prove that : $\frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \frac{\tan 8\theta}{\tan 2\theta}$.

Solution L.H.S.

$= \frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \frac{\frac{1}{\cos 8\theta} - 1}{\frac{1}{\cos 4\theta} - 1} = \frac{1-\cos 8\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{1-\cos 4\theta}$

$= \frac{2\sin^2 4\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{2\sin^2 2\theta} = \frac{(2\sin 4\theta\cos 4\theta)}{\cos 8\theta} \times \frac{\sin 4\theta}{2\sin^2 2\theta}$

$= \left(\frac{2\sin 4\theta\cos 4\theta}{\cos 8\theta}\right) \times \left(\frac{2\sin 2\theta\cos 2\theta}{2\sin^2 2\theta}\right)$

$= \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \left(\frac{\sin 8\theta}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right)$

$= \tan 8\theta \cot 2\theta = \frac{\tan 8\theta}{\tan 2\theta}$ = R.H.S.

$\left[\because\ 1 - \cos 8\theta = 2\sin^2\frac{8\theta}{2} = 2\sin^2 4\theta\right.$

and $\left.1 - \cos 4\theta = 2\sin^2\frac{4\theta}{2} = 2\sin^2 2\theta\right]$

Illustration 12 Prove that

$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) = \frac{1}{8}$

Solution We have $\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$

and $\cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$

$\therefore$ L.H.S.

$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$= \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$

$$= \frac{1}{4}\left[\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right)\right] \quad [\because\ 1-\cos\theta = 2\sin^2\frac{\theta}{2}]$$

$$= \frac{1}{4}\left[\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)\right] = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8} = \text{R.H.S.}$$

Illustration 13 Prove that

$$\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8} = \frac{3}{2}.$$

Solution We have $\frac{7\pi}{8} = \pi - \frac{\pi}{8}$ and $\frac{5\pi}{8} = \pi - \frac{3\pi}{8}$

$$\Rightarrow \cos\frac{7\pi}{8} = -\cos\frac{\pi}{8} \text{ and } \cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8}$$

$$\Rightarrow \cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8} \text{ and } \cos^4\frac{5\pi}{8} = \cos^4\frac{3\pi}{8}$$

$$\therefore \text{ L.H.S.} = 2\cos^4\frac{\pi}{8}+2\cos^4\frac{3\pi}{8}$$

$$= 2\left[\left(\cos^2\frac{\pi}{8}\right)^2+\left(\cos^2\frac{3\pi}{8}\right)^2\right]$$

$$= 2\left\{\frac{1+\cos\frac{\pi}{4}}{2}\right\}^2+\left\{\frac{1+\cos\frac{3\pi}{4}}{2}\right\}^2$$

$$\left[\because\ \frac{1+\cos 2\theta}{2} = \cos^2\theta\right]$$

$$= \frac{1}{2}\left\{\left(1+\cos\frac{\pi}{4}\right)^2+\left(1+\cos\frac{3\pi}{4}\right)^2\right\}^2$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2+\left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{2}+\sqrt{2}\right)+\left(1+\frac{1}{2}-\sqrt{2}\right)\right\} = \frac{3}{2} = \text{R.H.S.}$$

Illustration 14 If $\pi < x < 2\pi$, prove that

$$\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}} = \cot\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

Solution L.H.S.

$$\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}} = \frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}$$

$$= \frac{\sqrt{2}\left|\cos\frac{x}{2}\right|+\sqrt{2}\left|\sin x\frac{x}{2}\right|}{\sqrt{2}\left|\cos\frac{x}{2}\right|-\sqrt{2}\left|\sin\frac{x}{2}\right|}$$

$$\left[\because\ \pi < x < 2\pi,\ \therefore\ \frac{\pi}{2} < \frac{x}{2} < \pi\right]$$

$$= \frac{\left|\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}\right|-\left|\sin\frac{x}{2}\right|} = \frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}}$$

Thus, cos x/2 is negative and sin x/2 is positive.

Dividing numerator and denominator by sin x/2, we get

$$\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}} = \frac{\cot\frac{x}{2}-1}{\cot\frac{x}{2}+1} = \cot\left(\frac{x}{2}+\frac{\pi}{4}\right) = \text{R.H.S.}$$

Illustration 15 Prove that

$(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \tan 9°$

Solution We have $\cos 3x = 4\cos^3 x - 3\cos x$.

Hence, $4\cos^2 x - 3 = \frac{\cos 3x}{\cos x}$ for all $x \neq (2k+1)\cdot\frac{\pi}{2}, k \in Z$.

$$(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \frac{\cos 27^o}{\cos 9^o}\cdot\frac{\cos 81^o}{\cos 27^o}$$

$$= \frac{\cos 81^o}{\cos 9^o} = \frac{\sin 9^o}{\cos 9^o} = \tan 9^o$$

DPP 5

Total Marks 40 **Time 30 Minute**

Question Number 1 to 10. **Marking Scheme :** +4 for correct answer 0 in all other cases.

1. Prove that the following

(i) $\frac{1+\cos\theta-\sin\theta}{-1+\cos\theta+\sin\theta} = \cot\frac{\theta}{2}$

(ii) $\dfrac{1+\cos\theta-\sin\theta}{1-\cos\theta+\sin\theta}=\cot\dfrac{\theta}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$

or $\cot\dfrac{\theta}{2}(\sec\theta-\tan\theta)$

(iii) $\dfrac{\sec\theta-\tan\theta+1}{\sec\theta+\tan\theta+1}=\tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$

(iv) $\dfrac{\sec\theta+\tan\theta-1}{\sec\theta-\tan\theta+1}=\tan\dfrac{\theta}{2}\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$

(v) $\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\tan\left(\dfrac{\pi}{4}-\theta\right)=\cot\left(\dfrac{\pi}{4}+\theta\right)=\sqrt{\dfrac{1-\sin 2\theta}{1+\sin 2\theta}}$

(vi) $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan\left(\dfrac{\pi}{4}+\theta\right)=\cot\left(\dfrac{\pi}{4}-\theta\right)=\sqrt{\dfrac{1+\sin 2\theta}{1-\sin 2\theta}}$

2. Show that

(i) $\sin\theta\sin(60°-\theta)\sin(120°-\theta)=\dfrac{1}{4}\sin 3\theta$

(ii) $\cos\theta\cos(60°+\theta)\sin(120°+\theta)=-\dfrac{1}{4}\cos 3\theta$

(iii) $\sin\theta\sin(240°+\theta)\sin(120°+\theta)=-\dfrac{1}{4}\sin 3\theta$

(iv) $\cos\theta\cos(240°-\theta)\cos(120°-\theta)=\dfrac{1}{4}\cos 3\theta$

3. Prove that $\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\sec 2\theta-\tan 2\theta$

4. Prove that $\left(\dfrac{\pi}{4}+\theta\right)-\tan\left(\dfrac{\pi}{4}-\theta\right)=2\tan 2\theta$

5. Prove that : $\dfrac{1+\sin 2A-\cos 2A}{1+\sin 2A+\cos 2A}=\tan A$

6. Show that : $\dfrac{1}{\sin 10^{o}}-\dfrac{\sqrt{3}}{\cos 10^{o}}=4$

7. Prove that : $\dfrac{1+\sin 2A}{\cos 2A}=\dfrac{\cos A+\sin A}{\cos A-\sin A}=\tan\left(\dfrac{\pi}{4}+A\right)$

8. Prove that : If α and β are the two different roots of equation $a\cos\theta+b\sin\theta=c$, prove that

(i) $\tan(\alpha+\beta)=\dfrac{2ab}{a^2-b^2}$ (ii) $\cos(\alpha+\beta)=\dfrac{a^2-b^2}{a^2+b^2}$

9. If θ is an acute angle and $\sin\dfrac{\theta}{2}=\sqrt{\dfrac{x-1}{2x}}$, find $\tan\theta$ in terms of x.

10. Prove that $(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta)=\dfrac{\tan 8\theta}{\tan\theta}$

Result Analysis

1. 32 to 40 Marks : **Advance Level.**
2. 22 to 31 Marks : **Main Level.**
3. < 21 Marks : **Below Average**
   (Please go through this article again)

VALUES OF TRIGONOMETRIC RATIOS OF STANDARD ANGLES

1. Value of sin 15°, cos 15°, sin 75°, cos 75°, tan 15°, tan 75°

$\sin 15°=\sin(45°-30°)=\sin 45°\cos 30°-\sin 30°\cos 45°$

$$=\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}-\frac{1}{2}\frac{1}{\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

Also, $\sin 15°=\cos 75°=-\cos 105°$

Similarly, we can prove that $\cos 15°=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$

Also $\cos 15°=\sin 75°=\sin 105°$

$\tan 15°=\tan(60°-45°)$

$$=\frac{\tan 60^{o}-\tan 45^{o}}{1+\tan 60^{o}\tan 45^{o}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}$$

$\tan 75°=\tan(60°+45°)$

$$=\frac{\tan 60^{o}+\tan 45^{o}}{1-\tan 60^{o}\tan 45^{o}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=2+\sqrt{3}$$

2. Value of sin 18°, cos 18°

Let $\theta=18°$, then $5\theta=90°$

or $2\theta+3\theta=90°$

or $2\theta=90°-3\theta$

or $\sin 2\theta=\sin(90°-3\theta)$

or $\sin 2\theta=\cos 3\theta$ [Dividing by $\cos\theta$]

or $2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta$

or $2\sin\theta=4(1-\sin^2\theta)-3=1-4\sin^2\theta$

or $4\sin^2\theta+2\sin\theta-1=0$

or $\sin\theta=\dfrac{-2\pm\sqrt{4+16}}{8}=\dfrac{-2\pm 2\sqrt{5}}{8}=\dfrac{-1\pm\sqrt{5}}{4}$

$\because\ \theta=18°$

$\therefore$ $\sin\theta = \sin 18° > 0$, for 18° lies in the first quadrant.

$\therefore$ $\sin\theta$, i.e., $\sin 18° = \dfrac{\sqrt{5}-1}{4}$

Value of cos 18° :

$\cos^2 18° = 1 - \sin^2 18°$

$$= 1-\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1-\frac{5+1-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16}$$

$\Rightarrow$ $\cos 18^o = \dfrac{1}{4}\sqrt{10+2\sqrt{5}}$ $\quad [\because \cos 18^o > 0]$

3. Value of cos 36°, sin 36° :

$$\cos 36° = 1 - \sin^2 18° = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}$$

Value of sin 36° :

$$\sin^2 36^o = 1-\cos^2 36^o = 1-\left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$= 1-\frac{6+2\sqrt{5}}{16} = \frac{16-6-2\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16}$$

$\therefore$ $\sin 36^o = \dfrac{1}{4}\sqrt{10-2\sqrt{5}}$

NOTE

- $\sin 54° = \sin(90° - 36°) = \cos 36° = \dfrac{\sqrt{5}+1}{4}$
- $\cos 54° = \cos(90° - 36°) = \sin 36° = \dfrac{1}{4}\left(\sqrt{10-2\sqrt{5}}\right)$

4. Value of $\tan 7\dfrac{1^o}{2}, \cot 7\dfrac{1^o}{2}$

Let $\theta = 7\dfrac{1}{2}^o$, then $2\theta = 15^o$

$$\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}$$

$[\because 1 - \cos 2\theta = 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta]$

$$= \frac{1-\cos 15^o}{\sin 15^o} = \frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}-1\right)$$

Value of $\cot 82\dfrac{1^o}{2}$:

$$\cot 82\frac{1^o}{2} = \cot\left(90^o - 7\frac{1^o}{2}\right) = \tan 7\frac{1^o}{2}$$

$$= \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}-1\right)$$

Value of $\cot 7\dfrac{1^o}{2}$:

Let $\theta = 7\dfrac{1^o}{2}$, then $2\theta = 15°$

$$\text{Now, } \cot\theta = \frac{1+\cos 2\theta}{\sin 2\theta} = \frac{1+\cos 15^o}{\sin 15^o} = \frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1} = \left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}+1\right)$$

value of $\tan 82\dfrac{1^o}{2}$:

$$\tan 82\frac{1^o}{2} = \tan\left(90^o - 7\frac{1^o}{2}\right)$$

$$= \cot 7\frac{1^o}{2} = \left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}+1\right)$$

5. Value of $\theta = 22.5°$

We know that $\cos 2\theta = 2\cos^2\theta - 1$

$\therefore$ for $\theta = 22.5°$, we have $\cos 45° = 2\cos^2 22.5° - 1$

$\therefore$ $\cos 22.5° = \sqrt{\dfrac{1+\cos 45°}{2}}$

$$= \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\sin 22.5° = \sqrt{\frac{1-\cos 45°}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\tan 22.5° = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1$$

$$\cot 22.5° = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \sqrt{2}+1$$

All these values are tabulated as follows

	7.5^o	15^o	18^o	22.5^o	36^o	67.5^o	75^o
sin	$\frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\frac{\sqrt{8+2\sqrt{6}+2\sqrt{2}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}-1\right)$	$2-\sqrt{3}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}+1\right)$	$2+\sqrt{3}$	$\sqrt{\left(5+2\sqrt{5}\right)}$	$\sqrt{2}+1$	$\sqrt{\left(1+\frac{2}{\sqrt{5}}\right)}$	$\sqrt{2}-1$	$2-\sqrt{3}$

Illustration 1 Prove that : $4\left(\cos 66^o + \sin 84^o\right) = \sqrt{3} + \sqrt{15}$

Solution. : LHS $= 4(\cos 66° + \sin 84°)$

$= 4(\cos 66° + \cos 6°)$

$$= 4\left[2\cos\left(\frac{66^o + 6^o}{2}\right)\cos\left(\frac{66^o - 6^o}{2}\right)\right]$$

$$= 8\cdot\cos 36^o \cos 30^o = 8\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{15} + \sqrt{3} = \text{R.H.S.}$$

Illustration 2 Show that :

(i) $\sin 34° + \cos 64° - \cos 4° = 0$

(ii) $\sin 78° - \sin 18° + \cos 132° = 0$

(iii) $\cos\theta + \cos(120° + \theta) + \cos(240° + \theta) = 0$

(iv) $\cos 12° + \cos 84° + \cos 132° + \cos 156° = -\frac{1}{2}$

Solution. : (i) L.H.S. $= \sin 34° + \cos 64° - \cos 4°$

$$= \sin 34° - 2\sin\left(\frac{64^o + 4^o}{2}\right)\sin\left(\frac{64^o - 4^o}{2}\right)$$

$= \sin 34° - 2\sin 34° \sin 30°$

L.H.S. $= \sin 34° - \sin 34° = 0 =$ R.H.S.

(ii) $\sin 78° - \sin 18° + \cos 132° = 0$

$$= 2\cos\left(\frac{78^o + 18^o}{2}\right)\sin\left(\frac{78^o - 18^o}{2}\right) + \cos 132^o$$

$= 2\cos 48° \sin 30° + \cos 132°$

$= \cos 48° - \cos 48°$

$= 0 =$ R.H.S.

(iii) L.H.S.

$= \cos\theta + \cos(120 + \theta) + \cos(240 + \theta)$

$= \cos\theta + 2\cos(180 + \theta)\cos(60°)$

$= \cos\theta - \cos\theta$

$= 0 =$ R.H.S.

(iv) L.H.S.

$= \cos 12° + \cos 84° + \cos 132° + \cos 156°$

$= (\cos 132° + \cos 12°) + (\cos 156° + \cos 84°)$

$= 2\cos 72° \cos 60° + 2\cos 120° + \cos 36°$

$$= \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2} = \text{R.H.S.}$$

Illustration 3 Find the angle θ whose cosine is equal to its tangent.

Solution Given, $\cos\theta = \tan\theta$ or $\cos^2\theta = \sin\theta$

or $1 - \sin^2\theta = \sin\theta$ or $\sin^2\theta + \sin\theta - 1 = 0$

or $\sin\theta = \frac{-1\pm\sqrt{5}}{2} = 2\frac{\sqrt{5}-1}{4} = 2\sin 18^o$

or $\theta = \sin^{-1}(2\sin 18°)$

Illustration 4 Prove that $\cos 36° \cos 72° \cos 108° \cos 144° = 1/16$.

Solution $\cos 36° \cos 72° \cos 108° \cos 144°$

$= \cos 36° \sin 18° (-\sin 18°)(-\cos 36°)$

$$\cos^2 36° \sin^2 18° = \left(\frac{\sqrt{5}+1}{4}\right)^2\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= \left[\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)\right]^2 = \frac{1}{16}$$

Illustration 5 Prove that : $\frac{1-\tan^2\left(\frac{\pi}{4}-A\right)}{1+\tan^2\left(\frac{\pi}{4}-A\right)} = \sin 2A$.

Solution $\frac{1-\tan^2\left(\frac{\pi}{4}-A\right)}{1+\tan^2\left(\frac{\pi}{4}-A\right)} = \frac{1-\tan^2\theta}{1+\tan^2\theta}$ (where $\frac{\pi}{4} - A = \theta$)

$$= \cos 2\theta = \cos\left(\frac{\pi}{2} - 2A\right) = \sin 2A$$

TO FIND SUM OF SINES AND COSINES WHOSE ANGLES ARE IN ARITHMETIC PROGRESSION

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... + \sin(\alpha+(n-1)\beta)$$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Proof

Let

$S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... + \sin(\alpha+(n-1)\beta)$

Here angles are in A.P. and common difference of angles is β.

$$= \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \times \sin\left[\frac{\text{First angle} + \text{last angle}}{2}\right]$$

Multiplying both sides by 2 $\sin\frac{\beta}{2}$, we get

$$2\sin\frac{\beta}{2}S = 2\sin\frac{\beta}{2}\sin\alpha + 2\sin\frac{\beta}{2}\sin(\alpha+\beta)$$

$$+2\sin\frac{\beta}{2}\sin(\alpha+2\beta) + ... + 2\sin\frac{\beta}{2}\sin(a+(n-1)\beta)$$

$$= \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \left(n - \frac{1}{2}\right)\beta\right)$$

$$= 2\sin\left(\alpha + \frac{(n-1)}{2}\beta\right)\sin\frac{n\beta}{2}$$

$$\therefore \quad S = \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ...+\cos(a+(n-1)\beta)$$
$$= \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Deductions

1. $\sin\alpha + \sin(\alpha+\beta) + \tan(\alpha+2\beta) + ... + \sin(\alpha+(n-1)\beta) = 0$

$$\Leftrightarrow \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} = 0$$

$$\Leftrightarrow \sin\frac{n\beta}{2} = 0 \quad \Leftrightarrow \frac{n\beta}{2} = k\pi \Leftrightarrow \beta = \frac{2k\pi}{n}$$

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ... + \cos(\alpha + n - 1\beta) = 0$

$$\Leftrightarrow \beta = \frac{2k\pi}{n}$$

$\sin x + \sin 3x + \sin 5x + ... + \sin(2n-1)x$

$$= \frac{\sin\left(x + \frac{n-1}{2}2x\right)\sin\left(\frac{n}{2}\cdot 2x\right)}{\sin\frac{2x}{2}} \qquad (\text{Here } \alpha = x, \beta = 2x)$$

$$= \frac{\sin^2(nx)}{\sin x} \text{ or } \frac{1-\cos 2nx}{2\sin x} \qquad .....(i)$$

$$\cos A \cos 2A \cos 2^2A \cos 2^3 A .... \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$$

Proof

L.H.S. = $\cos A \cos 2A \cos 2^2A \cos 2^3A .... \cos 2^{n-1} A$

$$= \frac{1}{2\sin A}[(2\sin A\cos A)\cos 2A\cos 2^2 A .... \cos 2^{n-1} A]$$

$$= \frac{1}{2\sin A}[(\sin 2A\cos 2A)\cos 2^2 A ..... \cos 2^{n-1} A]$$

$$= \frac{1}{2^2\sin A}[(2\sin 2A\cos 2A)\cos 2^2A ..... \cos 2^{n-1} A]$$

$$= \frac{1}{2^2\sin A}[\sin 2(2A)\cdot\cos 2^2A ..... \cos 2^{n-1} A]$$

$$= \frac{1}{2^3\sin A}[(2\sin 2^2A\cos 2^2A) ..... \cos 2^{n-1} A]$$

$$= \frac{1}{2^3\sin A}[\sin(2\times 2^2 A) ..... \cos 2^{n-1} A]$$

$$= \frac{1}{2^3\sin A}[(\sin 2^3 A\cos 2^3A ..... \cos 2^{n-1} A]$$

.....

.....

$$= \frac{1}{2^{n-1}\sin A}[\sin 2^{n-1} A \cos 2^{n-1} A]$$

$$= \frac{1}{2^n\sin A}[2\sin 2^{n-1} A\cos 2^{n-1} A]$$

$$= \frac{1}{2^n\sin A}\sin(2\times 2^{n-1} A)$$

$$= \frac{1}{2^n\sin A}\sin 2^n A = \text{R.H.S}$$

Illustration 1 Find the value of

$$\sin\frac{\pi}{n}+\sin\frac{3\pi}{n}+\sin\frac{5\pi}{n}+....+\sin\frac{(2n-1)}{n}\pi$$

Solution Here $\alpha=\frac{\pi}{n}, \alpha+\beta=\frac{3\pi}{n} \Rightarrow \beta=\frac{2\pi}{n}$ where K = 1

$$=\frac{\sin\left(\frac{\pi}{n}+\frac{2^{n-2}}{2}\cdot\frac{2\pi}{n}\right)\sin\pi}{\sin\frac{\pi}{n}}=0$$

$\therefore$ Given sum = 0

Illustration 2 Evaluate : sin 1° + sin 3° + sin 5° + ...+ sin 89°.

Solution Here $\alpha = 1°$, $\beta = 2°$ and n = 45

Therefore, given sum

$$=\frac{\sin\left(1^o+\frac{45-1}{2}2^o\right)\sin\left(\frac{45.2}{2}\right)^o}{\sin\left(\frac{2}{2}\right)^o}$$

$$=\frac{\sin 45^o \sin 45^o}{\sin 1^o}=\frac{1}{2\sin 1^o}$$

Illustration 3 Sum the series to n terms :

$$\sin^2\theta+\sin^2\left(\theta+\frac{2\pi}{n}\right)+\sin^2\left(\theta+\frac{4\pi}{n}\right)+....+$$

$$\sin^2\left(\theta+2(n-1)\frac{\pi}{n}\right)$$

Solution The r^{th} term of the given series is given by

$$t_r=\sin^2\left(\theta+\frac{2(r-1)\pi}{n}\right)$$

$$=\frac{1}{2}\left[1-\cos\left(2\theta+\frac{4(r-1)\pi}{n}\right)\right]$$

$$\left(\because \sin^2 x=\frac{1-\cos 2x}{2x}\right)$$

$$S_n=\sum_{r=1}^{n}t_r=\frac{1}{2}\left\{\sum_{r=1}^{n}1-\sum_{r=1}^{n}\cos\left(2\theta+\frac{4(n-1)\pi}{n}\right)\right\}$$

$$=\frac{n}{2}-0 \quad \left(\because \text{which even multiple of } \frac{\pi}{n}\right)$$

$$S_n=\frac{n}{2}=\sum_{r=1}^{n}\cos^2\left(\theta+\frac{2(r-1)\pi}{n}\right)=\frac{n}{2}$$

Illustration 4 Show that $\sum_{r=1}^{n}\sin^4\left(\theta+\frac{2(r-1)\pi}{n}\right)=\frac{3n}{8}$ hence

deduce the value of $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$

Solution We know that $\sin^4 x=\frac{1}{8}[3-4\cos 2x+\cos 4x]$

$$S_n=\sum_{r=-1}^{n}\sin^4\left(\theta+\frac{2(r-1)\pi}{n}\right)$$

$$=\frac{1}{8}\sum_{r=-1}^{n}\left[3-4\cos\left(2\theta+\frac{4(r-1)\pi}{n}\right)+\cos\left(4\theta+\frac{8(r-1)\pi}{n}\right)\right]$$

We have $\sum_{r=-1}^{n}\cos\left(2\theta+\frac{4(r-1)\pi}{n}\right)=0$ where

$$\beta=\frac{4\pi}{n}=2\cdot\frac{2\pi}{n} \ \& \ \sum_{r=-1}^{n}\cos\left(2\theta+\frac{8(r-1)\pi}{n}\right)=0$$

where $\beta=\frac{8\pi}{n}=4\cdot\frac{2\pi}{n}$ Thus, $S_n=\frac{3n}{8}$

Deduction

Put $\theta=\frac{\pi}{8}, n=4$, we get

$$\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}$$

Illustration 5 Evaluate $\sum_{r=1}^{n}\cos^4\left(\theta+\frac{2(r-1)\pi}{n}\right)=\frac{3n}{8}$

Hence, deduce that

$$\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$$

Solution Use $\cos^4 x=\frac{1}{8}[3+4\cos 2x+\cos 4x]$

Illustration 6 The average of 2 sin 2°, 4 sin 4°, 6 sin 6°, ..., 180 sin 180° is :

(A) tan 1° (B) cot 1°

(C) cos 1° (D) sin 1°

Solution (B)

Let S = 2 sin2° + 4 sin 4° + 6 sin 6° + ...+ 178 sin 178° ....(i) write S in reverse order, we get

$S = 178 \sin 178° + 176 \sin 176° + 174 \sin 174° + ... + 2 \sin 2°$

$S = 178 \sin 2° + 176 \sin 4° + 174 \sin 6° + ... + 2 \sin 178°$ ......(ii)

Adding (i) and (ii),we get

$2S = 180 (\sin 2° + \sin 4° + \sin 6° + ... + \sin 178°)$

$$S = 90\left[\frac{\sin\left(2^o + \frac{89-1}{2}\cdot 2^o\right)\sin\left(\frac{89}{2}\cdot 2\right)^o}{\sin\left(\frac{2}{2}\right)^o}\right]$$

$$S = 90\left[\frac{\sin 90^o \sin 89^o}{\sin 1^o}\right]$$

$S = 90 \cot 1°$

Now average of given numbers

$$= \frac{S + 180\sin 180^o}{90} = \frac{90\cot 1^o}{90} = \cot 1^o$$

Illustration 7 Find the value of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$.

Solution $S = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

$$= \frac{\sin\left(3\frac{\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)}\cos\left(\frac{\pi}{7} + \frac{3\pi}{7}\right)$$

$$= \frac{2\sin\left(\frac{3\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = \frac{\sin\left(\frac{7\pi}{7}\right) - \sin\left(\frac{\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = -\frac{1}{2}$$

Illustration 8 Prove that :

$$\frac{\cos 3x}{\sin 2x \sin 4x} + \frac{\cos 5x}{\sin 4x \sin 6x} + \frac{\cos 7x}{\sin 6x \sin 8x} + \frac{\cos 9x}{\sin 8x \sin 10x}$$

$$\frac{1}{2}(\text{cosec } x)[\text{cosec } 2x - \text{cosec } 10x]$$

Solution Let $f(x) = \frac{\cos 3x}{\sin 2x \sin 4x} + \frac{\cos 5x}{\sin 4x \sin 6x}$

$$+ \frac{\cos 7x}{\sin 6x \sin 8x} + \frac{\cos 9x}{\sin 8x \sin 10x}$$

Multiple and divide by (2 sin x) in whole expression

$$f(x) = \frac{\sin 4x - \sin 2x}{2\sin x \sin 2x \sin 4x} + \frac{\sin 6x - \sin 4x}{2\sin x \sin 4x \sin 6x}$$

$$+ \frac{\sin 8x - \sin 6x}{2\sin x \sin 6x \sin 8x} + \frac{\sin 10x - \sin 8x}{2\sin x \sin 8x \sin 10x}$$

$= 1/2 \text{ cosec } x [\text{cosec } 2x - \text{cosec } 10x]$

Illustration 9 If $\theta = \frac{\pi}{2^n + 1}$, show that

$$\cos\theta \cos 2\theta\, 2\cos 2^2\theta \cdots \cos 2^{n-1}\theta = \frac{1}{2^n}.$$

Solution In the above result, put $\theta = \frac{\pi}{2^n + 1}$

$$\text{R.H.S.} = \frac{\sin 2^n\theta}{2^n \sin\theta} = \frac{\sin\left(\frac{\pi}{2^n+1}\right)2^n}{2^n \sin\left(\frac{\pi}{2^n+1}\right)} = \frac{\sin\left(\frac{2^n+1-1}{2^n+1}\right)\pi}{2^n \sin\left(\frac{\pi}{2^n+1}\right)}$$

$$= \frac{\sin\left(\pi - \frac{\pi}{2^n+1}\right)}{2^n \sin\left(\frac{\pi}{2^n+1}\right)} = \frac{\sin\left(\frac{\pi}{2^n+1}\right)}{2^n \sin\left(\frac{\pi}{2^n+1}\right)} = \frac{1}{2^n}$$

DPP 6

Total Marks 20 **Time 20 Minute**

Question Number 1 to 5. **Marking Scheme :** +4 for correct answer 0 in all other cases

1. Prove that $\sin^2 48° - \cos^2 12° = \frac{\sqrt{5}+1}{8}$.
2. Prove that $4(\sin 24° + \cos 6°) = \sqrt{3} + \sqrt{15}$.
3. Find the value of $\sin 47° + \sin 61° - \sin 11° - \sin 25°$
4. Find the value of

$$\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$$

5. Find the value of $\sum_{r=1}^{n-1} \sin^2\frac{r\pi}{n}$

Result Analysis

1. 16 to 20 Marks : **Advance Level.**
2. 12 to 15 Marks : **Main Level.**
3. < 12 Marks : **Below Average**
(Please go through this article again)

CONDITIONAL IDENTITIES

Some Standard Identities in Triangle

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Proof

In ΔABC, we have $A + B + C = \pi$

or $A + B = \pi - C$

or $\tan(A + B) = \tan(\pi - C)$

or $\dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

or $\tan A + \tan B$

$= -\tan C + \tan A \tan B \tan C$

or $\tan A + \tan B + \tan C$

$= \tan A \tan B \tan C$

2. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{C}{2}\tan\frac{B}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

Proof

Since $A + B + C = \pi$, we have $\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

or $\tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\frac{C}{2}$

or $\dfrac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \dfrac{1}{\tan\frac{C}{2}}$

or $\tan\frac{A}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2} = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$

or $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

3. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Proof

$(\sin 2A + \sin 2B) + \sin 2C$

$= 2\sin(A + B)\cos(A - B) + \sin 2C$

$= 2\sin(\pi - C)\cos(A - B) + \sin 2C$

$= 2\sin C\cos(A - B) + 2\sin C\cos C$

$= 2\sin C[\cos(A - B) + \cos C]$

$= 2\sin C[\cos(A - B) + \cos\{\pi - (A + B)\}]$

$= 2\sin C[\cos(A - B) - \cos(A + B)]$

$= 2\sin C \times 2\sin A\sin B = 4\sin A\sin B\sin C$

4. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

Proof

$(\cos 2A + \cos 2B) + \cos 2C$

$= 2\cos(A + B)\cos(A - B) + 2\cos^2 C - 1$

$= 2\cos(\pi - C)\cos(A - B) + 2\cos^2 C - 1$

$= -2\cos C\cos(A - B) + 2\cos^2 C - 1$

$= -2\cos C[\cos(A - B) - \cos C] - 1$

$= -2\cos C[\cos(A - B) - \cos\{\pi - (A + B)\}] - 1$

$= -2\cos C[\cos(A - B) + \cos(A + B)] - 1$

$= -1 - 4\cos A\cos B\cos C$

5. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Proof

$(\cos A + \cos B) + \cos C - 1$

$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos C - 1$

$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\frac{A-B}{2} + \cos C - 1$

$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} - 1$

$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 2\sin^2\frac{C}{2}$

$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right]$

$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right]$

$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\left(\frac{A+B}{2}\right)\right]$

$= 2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right) = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

6. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Proof

$(\sin A + \sin B) + \sin C = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin C$

$= 2\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\frac{A-B}{2} + \sin C$

$= 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\sin\frac{C}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right]$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

NOTE

tan A + tan B + tan C = tan A tan B tan C is true for A + B + C = nπ, where n ∈ N.

Illustration 1 If A + B + C = π then prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Solution L.H.S. = $\sin^2\frac{A}{2}+\left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)$

$$=1-\cos^2\frac{A}{2}+\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$$

$$=1-\cos^2\frac{A}{2}+\cos\frac{A}{2}\sin\left(\frac{B-C}{2}\right)$$

$$=1-\cos\frac{A}{2}\left\{\cos\frac{A}{2}-\sin\frac{B-C}{2}\right\}$$

$$=1-\cos\frac{A}{2}\left\{\sin\left(\frac{B+C}{2}\right)-\sin\left(\frac{B-C}{2}\right)\right\}$$

$$=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Illustration 2 If A + B + C = π, prove that

$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$.

Solution L.H.S. = $\cos^2 A + \cos^2 B + \cos^2 C$

$$=\frac{1}{2}[2\cos^2 A+2\cos^2 B+2\cos^2 C]$$

$$=\frac{1}{2}[(1+\cos 2A)+(1+\cos 2B)+(1+\cos 2C)]$$

$$=\frac{1}{2}[3+\cos 2A+\cos 2B+\cos 2C]$$

$$=\frac{1}{2}[3-1-4\cos A\cos B\cos C]$$

$$= 1 - 2\cos A\cos B\cos C$$

Illustration 3 If A + B + C + D = 360° then show that

sinA – sin B + sin C – sin D

$$= -4\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A+D}{2}\right)\sin\left(\frac{A+C}{2}\right)$$

Solution L.H.S.

$$=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)+2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)-2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$=2\cos\left(\frac{A+B}{2}\right)\left[\sin\left(\frac{A-B}{2}\right)-\sin\left(\frac{C-D}{2}\right)\right]$$

$$=2\cos\left(\frac{A+B}{2}\right)\left[2\cos\left(\frac{A-B+C-D}{4}\right)\right.$$

$$\left.\sin\left(\frac{A-B-C+D}{4}\right)\right]$$

$$=2\cos\left(\frac{A+B}{2}\right)\left[2\cos\left(\frac{A+C-2\pi+A+C}{4}\right)\right.$$

$$\left.\sin\left(\frac{A+D-C+D}{4}\right)\right]$$

$$=2\cos\left(\frac{A+B}{2}\right)\left[2\cos\left(\frac{\pi}{2}-\frac{A+C}{2}\right)\sin\left(\frac{A+D}{2}-\frac{\pi}{2}\right)\right]$$

$$=-4\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A+D}{2}\right)$$

Illustration 4 If x + y + z = xyz, prove that

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$$

Solution Let x = tan A, y = tan B, z = tan C

Now x + y + z = xyz

⇒ tan A + tan B + tan C = tan A tan B tan C

⇒ A + B + C = nπ

or 2A + 2B + 2C = 2nπ

or tan 2A + tan 2B + tan 2C = tan 2A tan 2B tan 2C or $\dfrac{2\tan A}{1-\tan^2 A}+\dfrac{2\tan B}{1-\tan^2 B}+\dfrac{2\tan C}{1-\tan^2 C}$

$$=\frac{2\tan A}{1-\tan^2 A}\frac{2\tan B}{1-\tan^2 B}\frac{2\tan C}{1-\tan^2 C}$$

$$\Rightarrow \frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$$

Illustration 5 In any triangle ABC, prove that

$\sin^3 A\cos(B-C)+\sin^3 B\cos(C-A)+\sin^3 C\cos(A-B)$
$=3\sin A\sin B\sin C$

Solution LHS $=\sin^2 A\sin(B+C)\cos(B-C)+\sin^2 B\sin(C+A)\cos(C-A)+\sin^2 C\sin(A+B)\cos(A-B)$

$$=\frac{1}{2}\sin^2 A(\sin 2B+\sin 2C)+\frac{1}{2}\sin^2 B$$

$$(\sin 2C+\sin 2A)+\frac{1}{2}\sin^2 C(\sin 2A+\sin 2B)$$

$=\sin^2 A(\sin B\cos B+\sin C\cos C)+\sin^2 B(\sin C\cos C+\sin A\cos A)+\sin^2 C(\sin A\cos A+\sin B\cos B)$

$=\sin A\sin B(\sin A\cos B+\cos A\sin B)+\sin B\sin C(\sin B\cos C+\cos B\sin C)+\sin C\sin A(\sin A\cos C+\cos A\sin C)$

$=\sin A\sin B\sin(A+B)+\sin B\sin C\sin(B+C)$
$+\sin C\sin A\sin(A+C)$
$=3\sin A\sin B\sin C=$ RHS.

Illustration 6 The product of the sines of the angles of a triangle is p and the product of their cosines is q. Show that the tangents of the angles are the roots of the equation of $qx^3-px^2+(1+q)x-p=0$.

Solution From the question

$\sin A\sin B\sin C=p$ and $\cos A\cos B\cos C=q$

$\therefore\ \tan A\tan B\tan C=p/q$

Also, $\tan A+\tan B+\tan C=\tan A\tan B\tan C=p/q$

Now, $\tan A\tan B+\tan B\tan C+\tan C\tan A$

$$=\frac{\sin A\sin B\cos C+\sin B\sin C\cos A+\sin C\sin A\cos B}{\cos A\cos B\cos C}$$

$$=\frac{1}{2q}[(\sin^2 A+\sin^2 B-\sin^2 C)+(\sin^2 B+\sin^2 C-\sin^2 A)$$
$$+(\sin^2 C+\sin^2 A-\sin^2 B)]$$

[$\because A+B+C=\pi$ and $2\sin A\sin B\cos C=\sin^2 A+\sin^2 B-\sin^2 C$]

$$=\frac{1}{2q}[\sin^2 A+\sin^2 B+\sin^2 C]$$

$$=\frac{1}{4q}[3-(\cos 2A+\cos 2B+\cos 2C)]$$

$$=\frac{1}{q}[1+\cos A\cos B\cos C]=\frac{1}{q}(1+q)$$

The equation whose roots are tan A, tan B tan C will be given by

$x^3-(\tan A+\tan B+\tan C)x^2+(\tan A\tan B+\tan B\tan C+\tan C\tan A)x-\tan A\tan B\tan C=0$

or $x^3-\dfrac{p}{q}x^2+\dfrac{1+q}{q}x-\dfrac{p}{q}=0,$

or $qx^3-px^2+(1+q)x-p=0$

Illustration 7 In a triangle ABC, if $\cos A\cos B\cos C=\dfrac{\sqrt{3}-1}{8}$,

$\sin A\sin B\sin C=\dfrac{3+\sqrt{3}}{8}$ then

(A) Cubic equation in 'x' whose roots are tan A, tan B tan C is $x^3-(3+2\sqrt{3})x^2+(5+4\sqrt{3})x-(3+2\sqrt{3})=0$

(B) Cubic equation in 'x' whose roots are tan A, tan B tan C is $x^3-(3+2\sqrt{3})x^2+(5+2\sqrt{3})x-(3+2\sqrt{3})=0$

(C) In the same triangle ABC, value of $\cos A+\cos B+\cos C=\dfrac{\sqrt{1}+\sqrt{2}+\sqrt{3}}{2\sqrt{2}}$

(D) In the same triangle ABC, value of $\cos A+\cos B+\cos C=\dfrac{\sqrt{2}+\sqrt{3}+3}{2\sqrt{2}}$

Solution $\cos A\cos B\cos C=\dfrac{\sqrt{3}-1}{8}$, $\sin A\sin B\sin C=\dfrac{3+\sqrt{3}}{8}$

$$\Rightarrow \tan A+\tan B+\tan C=\frac{3+\sqrt{3}}{\sqrt{3}-1}=\tan A\tan B\tan C$$

$\cos(A+B+C)=\cos A\cos B\cos C(1-S_2)$

$$=\frac{\sqrt{3}-1}{8}(1-S_2)$$

$\Rightarrow S_2=\tan A\tan B+\tan B\tan C+\tan C\tan A$

$$=1+\frac{8}{\sqrt{3}-1}=5+4\sqrt{3}$$

Equation with roots are tan A, tan B, tan C is

$$x^3 - (3+2\sqrt{3})x^2 + (5+4\sqrt{3})x - (3+2\sqrt{3}) = 0$$

Solving $x = 1, \sqrt{3}, 2+\sqrt{3}$

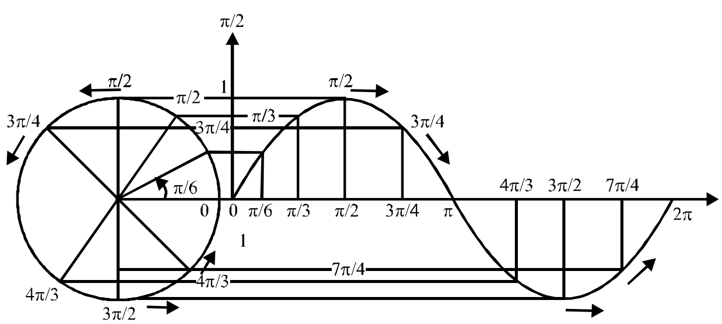
i.e., $\tan A = 1, \tan B = \sqrt{3}, \tan C = 2+\sqrt{3}$

$\Rightarrow$ $A° = 45°$; $B° = 60°$; $C° = 75°$

Now $\cos 45° + \cos 60° + \cos 75° = \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2}$

Hence option (A) & (C) are correct.

Graph of Sine Function with Its Varying Values of Unit Circle



GRAPH AND OTHER USEFUL DATA OF TRIGONOMETRIC FUNCTION

1. y = f(x) = sin x

Domain $\to$ R, Range $\to [-1, 1]$

Period $\to 2\pi$

$\sin^2 x, |\sin x| \in [0, 1]$

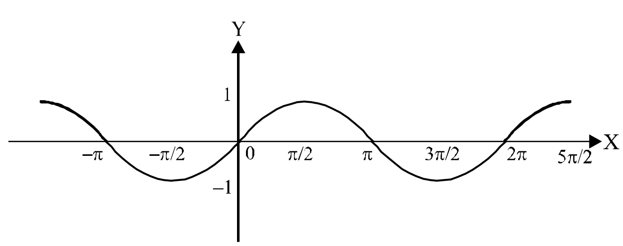
$\sin x = 0 \Rightarrow x = n\pi, n \in I$

$\sin x = 1 \Rightarrow x = (4n+1)\pi/2, n \in I$

$\sin x = -1 \Rightarrow x = (4n-1)\pi/2, n \in I$

$\sin = \sin\alpha \Rightarrow x = n\pi + (-1)^n \alpha, n \in I$

$\sin x \ge 0 \Rightarrow x \in \bigcup_{n\in I} [2n\pi, \pi + 2n\pi]$



2. y = f(x) = cos x

Domain $\to$ R, Range $\to [-1, 1]$

Period $\to 2\pi$

$\cos^2 x, |\cos x| \in [0, 1]$

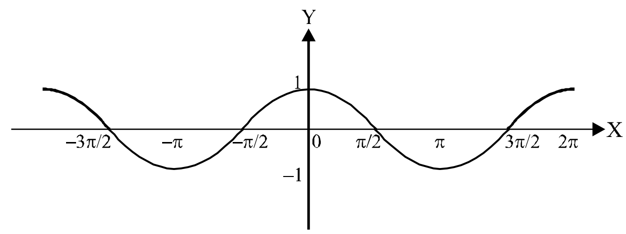
$\cos x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$

$\cos x = 1 \Rightarrow x = 2n\pi, n \in I$

$\cos x = -1 \Rightarrow x = (2n+1)\pi, n \in I$

$\cos x = \cos\alpha \Rightarrow x = 2n\pi \pm \alpha, n \in I$

$\cos x \ge 0 \Rightarrow x \in \bigcup_{n\in I} [2n\pi - \frac{\pi}{2}, 2n\pi + \pi/2]$



3. y = f(x) = tan x

Domain $\to R \sim (2n+1)\pi/2, n \in I$

Range $\to (-\infty, \infty)$

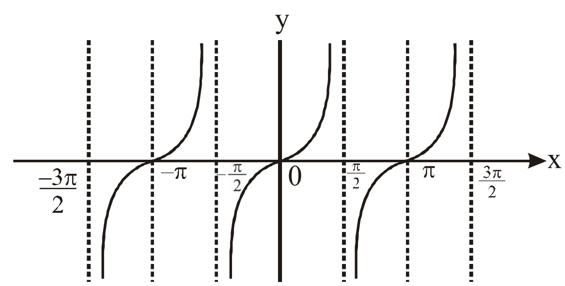
Period $\to \pi$

Discontinuous at $x = (2n+1)\pi/2, n \in I$

$\tan^2 x, |\tan x| \in [0,\infty)$

$\tan x = 0 \Rightarrow x = n\pi, n \in I$

$\tan x = \tan\alpha \Rightarrow x = n\pi + \alpha, n \in I$



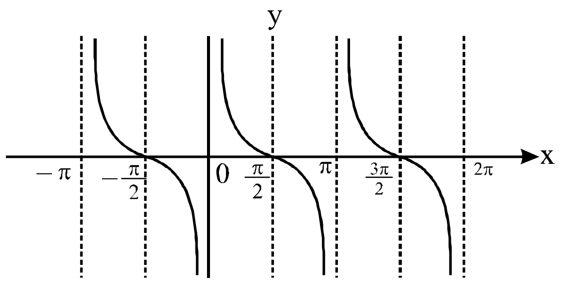
4. y = f(x) = cot x

Domain $\to R \sim n\pi, n \in I$; Range $\to (-\infty, \infty)$; Period $\to \pi$;

Discontinuous at $x = n\pi, n \in I$

$\cot^2 x, |\cot x| \in [0, \infty)$

$\cot x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$

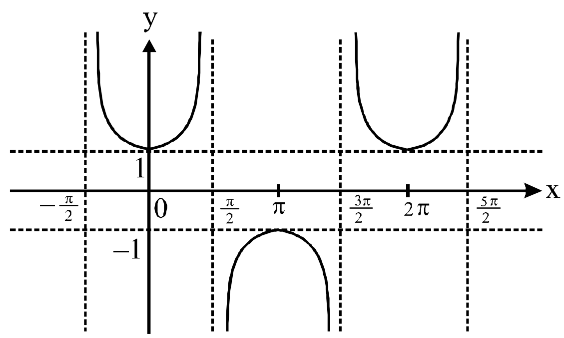


5. $y = f(x) = \sec x$

Domain $\to R \sim (2n+1)\,\pi/2,\ n \in I$;

Range $\to (-\infty,-1] \cup [1,\infty)$

Period $\to 2\pi$, $\sec^2 x$, $|\sec x| \in [1, \infty)$



6. $y = f(x) = \operatorname{cosec} x$

Domain $\to R \sim n\pi,\ n \in I$;

Range $\to (-\infty,-1] \cup [1,\infty)$

Period $\to 2\pi$, $\operatorname{cosec}^2 x$, $|\operatorname{cosec} x| \in [1, \infty)$

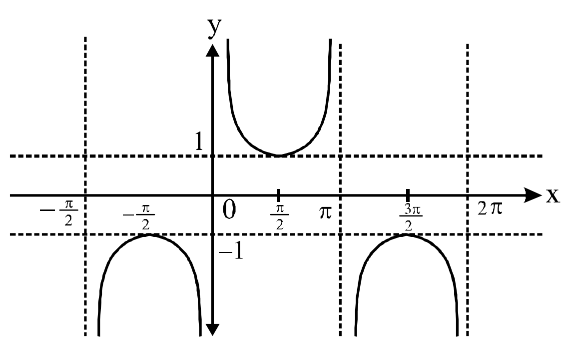


Illustration 1 Find the range of $f(x) = \dfrac{1}{4\cos x - 3}$

Solution $-1 \le \cos x \le 1$

$\Rightarrow\ -4 \le \cos x \le 4$

$\Rightarrow\ -7 \le 4\cos x - 3 \le 1$

$\Rightarrow\ -7 \le 4\cos x - 3 < 0$

$\Rightarrow\ 0 < 4\cos x - 3 \le 1 \qquad (\because\ 4\cos x - 3 \ne 0)$

$\Rightarrow\ -\dfrac{1}{7} \ge \dfrac{1}{4\cos x - 3} > -\infty \quad$ or $\quad \infty > \dfrac{1}{4\cos x - 3} \ge 1$

$\Rightarrow\ \dfrac{1}{4\cos x - 3} \in \left(-\infty, -\dfrac{1}{7}\right) \cup [1,\infty]$

Illustration 2 Find the range of $f(x) = \dfrac{1}{5\sin x - 6}$.

Solution $-1 \le \sin x \le 1$

or $\ -5 \le 5\sin x \le 5$

or $\ -11 \le 5\sin x - 6 \le -1$

or $\ -1 \le \dfrac{1}{5\sin x - 6} \le -1/11$

or $\ \dfrac{1}{5\sin x - 6} \in [-1, -1/11]$

Illustration 3 Find the range of $f(x) = \cos^2 x + \sec^2 x$.

Solution We have $f(x) = \cos^2 x + \sec^2 x$

$= (\cos x - \sec x)^2 + 2\cos x \sec x$

$= 2 + (\cos x - \sec x)^2 \ge 2$

Illustration 4 Find the range of $f(x) = \sin^2 x - 3\sin x + 2$.

Solution $f(x) = \sin^2 x - 3\sin x + 2$

$= (\sin x - 3/2)^2 + 2 - 9/4$

$= (\sin x - 3/2)^2 - 1/4$

Now $-1 \le \sin x \le 1$

or $\ -5/2 \le \sin x - 3/2 \le -1/2$

or $\ 1/4 \le (\sin x - 3/2)^2 \le 25/4$

or $\ 0 \le (\sin x - 3/2)^2 - 1/4 \le 6$

Illustration 5 Find the range of

$f(x) = \sqrt{\sin^2 x - 6\sin x + 9} + 3$

Solution $f(x) = \sqrt{\sin^2 x - 6\sin x + 9} + 3$

$= \sqrt{(\sin x - 3)^2} + 3$

$= |\sin x - 3| + 3$

Now $-1 \le \sin x \le 1$

or $\ -4 \le \sin x - 3 \le -2$

or $\ 2 \le |\sin x - 3| \le 4$

or $\ 5 \le |\sin x - 3| + 3 \le 7$

Illustration 6 Find the range of $f(x) = \operatorname{cosec}^2 x + 25\sec^2 x$.

Solution $f(x) = (1 + \cot^2 x) + 25(1 + \tan^2 x)$

$= 26 + \cot^2 x + 25\tan^2 x$

$= 26 + 10 + (\cot^2 x + 25\tan^2 x - 2\cot x\, 5\tan x)$

$= 36 + (\cot x - 5\tan x)^2 \ge 36$

Range of f(θ) = a cos θ + b sinθ

Let $a = r\sin\alpha$ and $b = r\cos\alpha$, then $r^2 = a^2 + b^2$ and $\tan\alpha = a/b$.

Now, $f(\theta) = a\cos\theta + b\sin\theta = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$

$= r\sin(\theta+\alpha) = \sqrt{a^2+b^2}\sin\left(\theta + \tan^{-1}\frac{a}{b}\right)$

Now, $-1 \le \sin\left(\theta + \tan^{-1}\frac{a}{b}\right) \le 1$

$\Rightarrow \quad -\sqrt{a^2+b^2} \le \sqrt{a^2+b^2}\sin\left(\theta+\tan^{-1}\frac{a}{b}\right) \le \sqrt{a^2+b^2}$

Hence, range is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$.

Illustration 1 Find the maximum value of $\sqrt{3}\sin x + \cos x$ and x for which a maximum value occurs.

Solution

$$\sqrt{3}\sin x + \cos x = 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\sin x\right) = 2\sin\left(x+\frac{\pi}{6}\right)$$

which is maximum when $x + \pi/6 = \pi/2$ or $x = 60^o$ and has a maximum value 2.

Illustration 2 Find the maximum and minimum values of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

Solution $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$

$$= \frac{1+\cos 2\theta}{2} - 3\sin 2\theta + 3\frac{(1-\cos 2\theta)}{2} + 2$$

$= 4 - \cos 2\theta - 3\sin 2\theta$

Now, $-\cos 2\theta - 3\sin 2\theta \in \left[-\sqrt{10}, \sqrt{10}\right]$

$\Rightarrow \quad 4 - \cos 2\theta - 3\sin 2\theta \in \left[4-\sqrt{10}, 4+\sqrt{10}\right]$

Illustration 3 For all θ in $\left[0, \frac{\pi}{2}\right]$ show that cos (sin θ) > sin (cos θ).

Solution $\theta \in \left[0, \frac{\pi}{2}\right]$

Now, $\quad \cos(\sin\theta) > \sin(\cos\theta)$

$\Rightarrow \quad \sin(\pi/2 - \sin\theta) > \sin(\cos\theta)$

Now we have to prove $\frac{\pi}{2} - \sin\theta > \cos\theta \; \forall \theta \in [0, \pi/2]$

Now, $f(\theta) = \pi/2 - \sin\theta - \cos\theta$ be a function. Then

$$f(\theta) = \frac{\pi}{2} - \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)$$

$$f(\theta) = \frac{\pi}{2} - \sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right)$$

$\because \quad 0 \le \theta \le \frac{\pi}{2}$

$\therefore \quad \frac{\pi}{4} \le \left(\theta + \frac{\pi}{4}\right) \le \frac{3\pi}{4}$

or $\quad \frac{1}{\sqrt{2}} \le \sin\left(\theta+\frac{\pi}{4}\right) \le 1$

or $\quad f(\theta) = \frac{\pi}{2} - \sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right) > 0 \;\; \forall \theta \in \left[0, \frac{\pi}{2}\right]$

or $\quad \frac{\pi}{2} - \sin\theta - \cos\theta > 0$

or $\quad \frac{\pi}{2} - \sin\theta > \cos\theta$

or $\quad \sin\left(\frac{\pi}{2} - \sin\theta\right) > \sin(\cos\theta)$

or $\quad \cos(\sin\theta) > \sin(\cos\theta) \; \forall \theta \in \left[0, \frac{\pi}{2}\right]$

Illustration 4 If $\sin^2(\theta-\alpha)\cos\alpha = \cos^2(\theta-\alpha)\sin\alpha = m\sin\alpha\cos\alpha$, then prove that $|m| \ge \frac{1}{\sqrt{2}}$.

Solution $\sin^2(\theta-\alpha)\cos\alpha = \cos^2(\theta-\alpha)\sin\alpha = m\sin\alpha\cos\alpha$

or $\quad \sin^2(\theta-\alpha) = m\sin\alpha$

$\cos^2(\theta-\alpha) = m\cos\alpha$

Adding, we get

$1 = m(\sin\alpha + \cos\alpha)$

or $\quad \sin\alpha + \cos\alpha = \frac{1}{m}$

or $\quad \sin\left(\alpha + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}\,m}$

or $\quad \left|\frac{1}{m\sqrt{2}}\right| \le 1$ or $|m| \ge \frac{1}{\sqrt{2}}$

Illustration 5 $y = \sin^2\left(\frac{15\pi}{8} - 4x\right) - \sin^2\left(\frac{17\pi}{8} - 4x\right)$.

Find range of y.

Solution We have

$$y = \sin^2\left(\frac{15\pi}{8} - 4x\right) - \sin^2\left(\frac{17\pi}{8} - 4x\right)$$

$\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$

$y = \sin(4\pi - 8x)\sin\left(-\frac{\pi}{4}\right)$

$y = \sin 8x \sin\frac{\pi}{4}$ $\qquad [\because \sin(4\pi - 8) = -\sin\theta]$

$y = \frac{\sin 8x}{\sqrt{2}}$

$y \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Illustration 6 $y = \frac{17 + 5\sin x + 12\cos x}{17 - 5\sin x - 12\cos x}$

Find min. and max. value of y.

Solution We have,

$y = \frac{17 + 5\sin x + 12\cos x}{17 - 5\sin x - 12\cos x}$ $\quad -13 \le 5\sin x + 12\cos x \le 13$

$y = \frac{17 + [-13, 13]}{17 - [-13, 13]}$

If y_{max} then value of denominator is min and numerator is max.

$y_{max} = \frac{30}{4} = \frac{15}{2}$

If y_{min} then the value of denominator is max and numerator is min.

$y_{min} = \frac{4}{30} = \frac{2}{15}$

Illustration 7 $y = \log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right)$.

Find range/ min. and max. value of y.

Solution $y = \log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right)$.

$y \log_2\left(\frac{[-5,5] + 15}{10}\right)$

$y_{max} = \log_2\left(\frac{20}{10}\right) = \log_2 2 = 1$

$y_{min} = \log_2\left(\frac{10}{10}\right) = \log_2 1 = 0$

Illustration 8 $y = \cos 2x + 3\sin x$. Find range, min. and max value of y.

Solution $y = \cos 2x + 3\sin x$

$y = 1 - 2\sin^2 x + 3\sin x$

$y = 1 - 2(\sin^2 x - \frac{3}{2}\sin x)$

$\Rightarrow \; y = 1 - 2\left(\sin^2 x - \frac{3}{2}\sin x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right)$

$\Rightarrow \; y = 1 - 2\left(\left(\sin x - \frac{3}{4}\right)^2 - \frac{9}{16}\right)$

$\Rightarrow \; y = 1 + \frac{9}{8} - 2\left(\sin x - \frac{3}{4}\right)^2$

$\Rightarrow \; y = \frac{17}{8} - 2\left(\sin x - \frac{3}{4}\right)^2$

$\Rightarrow \; y = \frac{17}{8} - 2\left([-1, 1]\frac{-3}{2}\right)^2$

$\Rightarrow \; y = \frac{17}{8} - 2\left(\frac{-7}{4}, \frac{1}{4}\right)^2$

$\Rightarrow \; y = \frac{17}{8} - 2\left[0, \frac{49}{16}\right]$

$\Rightarrow \; y = \frac{17}{8} - \left[0, \frac{49}{8}\right]$

$y_{max} = \frac{17}{8}$

$y_{min} = \frac{17 - 48}{8} = -4$

Illustration 9 $y = \cos^2 x - 4\cos x + 13$. Find min. and max value of y.

Solution $y = \cos^2 x - 4\cos x + 13$

$y = (\cos x - 2)^2 - 4 + 13$

$y = (\cos x - 2)^2 + 9$

$y = ([-1, 1] - 2)^2 + 9$

$y = ([-3, -1])^2 + 9$

$y = [1, 9] + 9$

$y_{min} = 10; \;\; y_{max} = 18$

Illustration 10 $y = a^2\tan^2 x + b^2\cot^2 x$ $(a, b \ge 0)$. find y_{min}.

Solution $y = a^2\tan^2 x + b^2\cot^2 x$

We know that AM ≥ GM

$$\frac{a^2\tan^2 + b^2\cot^2 x}{2} \geq \sqrt{a^2 + \tan^2 b^2 \cot^2 x}$$

$$\Rightarrow \frac{a^2\tan^2 x + b^2\cot^2 x}{2} \geq ab$$

$\Rightarrow$ $a^2\tan^2 x + b^2\cot^2 x \geq 2ab$

$y_{min} = ab$

Illustration 11 $y = 4\sin^2 x + \text{cosec}^2 x$, find y_{min}.

Solution $y = 4\sin^2 x + \text{cosec}^2 x$

We know that AM ≥ GM

$$\frac{4\sin^2 x + \text{cosec}^2 x}{2} \geq \sqrt{4\sin^2 x \times \frac{1}{\sin^2 x}}$$

$\Rightarrow$ $4\sin^2 x + \text{cosec}^2 x \geq 4$

$\Rightarrow$ $y_{min} = 4$

Illustration 12 $y = 18\sec^2 x + 8\cos^2 x$, find y_{min}.

Solution $y = 18\sec^2 x + 8\cos^2 x$

We know that AM ≥ GM

$$\frac{18\sec^2 x + 8\cos^2 x}{2} \geq \sqrt{18\sec^2 x \cdot 8\cos^2 x}$$

$$\Rightarrow \frac{18\sec^2 x + 8\cos^2 x}{2} \geq 12$$

$\Rightarrow$ $18\sec^2 x + 8\cos^2 x \geq 24$

$\Rightarrow$ $y_{min} = 24$

General or Miscellaneous problems

Parametric form of Circle & Ellipse

(i) $x^2 + y^2 = r^2 \quad \Rightarrow \quad x = r\cos\theta,\ y = r\sin\theta$

$= r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$= r^2$

(ii) $(x - h)^2 + (y - k)^2 = r^2$

$x = h + r\cos\theta,\ y = k + r\sin\theta$

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x = a\cos\theta \quad y = b\sin\theta$

Illustration 1 $x^2 + y^2 = 4$ and $a^2 + b^2 = 8$

find (ax + by) is minimum and maximum value.

Solution In finding range sometimes parametric co-ordinates can be used

$x^2 + y^2 = 4 \quad \Rightarrow \quad x = 2\cos\theta, \quad y = 2\sin\theta$

$a^2 + b^2 = 8 \quad \Rightarrow \quad a = 2\sqrt{2}\cos\theta, \quad y = 2\sqrt{2}\sin\theta$

$ax + by = 4\sqrt{2}\cos\theta\cos\phi + 4\sqrt{2}\sin\theta\sin\phi$

$ax + by = 4\sqrt{2}\cos(\theta - \phi)$

$ax + by|_{max} = 4\sqrt{2}, \qquad ax + by|_{min} = -4\sqrt{2}$

IMPORTANT INEQUALITIES

Illustration 1 In ΔABC, $\tan A + \tan B + \tan C \geq 3\sqrt{3}$, where A, B, C are acute angles.

Solution In ΔABC,

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Also, $\frac{\tan A + \tan B + \tan C}{3} \geq \sqrt[3]{\tan A \tan B \tan C}$

[since A.M. ≥ G.M.]

or $\tan A \tan B \tan C \geq \sqrt[3]{\tan A \tan B \tan C}$

or $\tan^2 A \tan^2 B \tan^2 C \geq 27$

or $\tan A \tan B \tan C \geq 3\sqrt{3}$,

or $\tan A + \tan B + \tan C \geq 3\sqrt{3}$, [cubing both sides]

Illustration 2 In ΔABC, prove that $\cos A + \cos B + \cos C \leq 3/2$.

Solution Let $\cos A + \cos B + \cos C = x$

$$\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2} = x$$

$$\text{or} \quad 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2} = x$$

$$\text{or} \quad 2\sin^2\frac{C}{2} - 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + x - 1 = 0$$

This is quadratic in sin C/2 which is real. So, discriminant $D \geq 0$.

$$4\cos^2\left(\frac{A-B}{2}\right) - 4 \times 2(x-1) \geq 0$$

$$\Rightarrow 2(x-1) \leq \cos^2\left(\frac{A-B}{2}\right)$$

or $2(x-1) \le 1$ or $x \le 3/2$

Thus, $\cos A + \cos B + \cos C \le \frac{3}{2}$

Illustration 3 If A, B,C are the angles of a triangle, then prove that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

hence prove the following

(i) $\sin\frac{A}{2}\sin^2\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$

(ii) $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\ge\frac{3}{4}$

Solution We have $A+B+C=\pi \Rightarrow \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$

Now, $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$

$= 1-\cos^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$

$= 1-\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin^2\frac{C}{2}$

$= 1-\sin\frac{C}{2}\left[\cos\frac{A+B}{2}-\cos\frac{A+B}{2}\right]$

$= 1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(i) Let $x=\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

then $\sum\sin^2\frac{A}{2}=1-2x$

Now, by AM – GM is inequally

$$\frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}}{3}\ge\sqrt[3]{\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}$$

$\Rightarrow$ $1-2x\ge 3(x)^{2/3} \Rightarrow (1-2x)^3\ge 27x^2$

$\Rightarrow$ $1-6x+12x^2-8x^3\ge 27x^2$

$\Rightarrow$ $8x^3+15x^2+6x-1\le 0$

$\Rightarrow$ $(x-1)^2(8x-1)\le 0 \Rightarrow 8x-1\le 0 \Rightarrow x\ge\frac{1}{8}$

Therefore, $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$

$\Leftrightarrow$ triangle is equilateral for x to be maximum.

DPP 7

Total Marks 40 **Time 30 Minute**

Question Number 1 to 10. **Marking Scheme :** +4 for correct answer 0 in all other cases.

1. In triangle ABC, prove that

(i) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(ii) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

2. If $A+B+C=\pi/2$, show that

(i) $\cot A+\cot B+\cot C=\cot A\cot B\cot C$

(ii) $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$

3. If $A+B+C=\pi$, prove that

$$\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin C\sin A}+\frac{\cos C}{\sin A\sin B}=2.$$

4. Find the range of $f(x)=\sqrt{4-\sqrt{1+\tan^2 x}}$

5. Find the range of $f(x)=\frac{1}{2|\cos x|-3}$

6. Find the range of $f(x)=\cos^4 x+\sin^2 x-1$.

7. Prove that

$(\sin\theta+\text{cosec}\,\theta)^2+(\cos\theta+\sec\theta)^2\ge 9$

8. Find the maximum value of $1+\sin\left(\frac{\pi}{4}+\theta\right)+2\sin\left(\frac{\pi}{4}-\theta\right)$ for all real values of θ.

9. Find the maximum and minimum values of $6\sin x\cos x+4\cos 2x$.

10. If $p(x)=\sin x(\sin^3 x+3)+\cos x(\cos^3 x+4)+1/2\sin^2 2x+5$, then find the range of $p(x)$.

Result Analysis

1. 32 to 40 Marks : **Advance Level.**

2. 22 to 31 Marks : **Main Level.**

3. < 22 Marks : **Below Average**
(Please go through this article again)

ILLUSTRATION

Illustration 1 The number of solutions of equation

$$\sum_{n=1}^{9}\cos(2n-1)\theta = \frac{1}{2}\text{in}\left(0,\frac{\pi}{4}\right) \text{ is equal to :}$$

(A) 2 (B) 3 (C) 4 (D) 5

Solution (C)

We have

$$\frac{\cos\theta+\cos3\theta+\cos5\theta+...+\cos7\theta}{\text{cosine series}}=\frac{1}{2}$$

$$\Rightarrow \frac{\cos(9\theta).\sin(9\theta)}{\sin(\theta)}=\frac{1}{2} \Rightarrow \sin(18\theta)=\sin\theta$$

$$\therefore \underbrace{18\theta = n\pi+(-1)^n\theta, n\in I}_{\text{when } n=2m, m\in I \qquad \text{when } n=2m+1}$$

$$\therefore \overbrace{18\theta = n\pi+(-1)^n\theta, n\in I}^{\text{when } n=2m, m\in I \qquad \text{when } n=2m+1}$$

$$\Rightarrow \theta=\frac{2m\pi}{17} \qquad \Rightarrow \theta=(2m+1)\frac{\pi}{19}$$

$$\therefore \theta=\frac{2\pi}{17},\frac{4\pi}{17} \text{ or } \theta=\frac{\pi}{19},\frac{3\pi}{19}$$

Illustration 2 The value of $(\cos^4 1°+\cos^4 2°+\cos^4 3°+...+\cos^4 179°)-(\sin^4 1°+\sin^4 2°+\sin^4 3°+....\sin^4 179°)$ equals.

(A) 2 cos 1° (B) –1

(C) 2 sin 1° (D) 0

Solution. (B)

Expression

$(\cos^4 1°+\cos^4 2°+\cos^4 3°+....+\cos^4 179°)$

$-(\sin^4 1°+\sin^4 2°+\sin^4 3°+....+\sin^4 179°)$

$=\cos 2°+\cos 4°+\cos 6°+....+\cos(358°)$

$$=\cos\frac{\left(\frac{2°+358°}{2}\right).\sin(179\times1°)}{\sin 1°}=\cos(180°)=-1$$

Illustration 3 If $x=2^{\log_b 3}$ and $y=3^{\log_b 2}$ then the value of $\log_2(\cos(x-y)+\sin(x-y)+\sec(x-y))$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Solution (A)

$x=2^{\log_b 3}$ and $y=3^{\log_b 2}=y$

$\therefore x-y=0$

Hence $\log_2(\cos(x-y)+\sin(x-y+\sec(x-y))$

$=\log_2(2)=1$

Illustration 4 The value of tan 9° + tan 36° + tan 9° · tan 36° is equal to:

(A) 2 (B) 1 (C) tan 60° (D) tan 30°

Solution (B)

$\tan 45° = \tan(9°+36°)$

$$\Rightarrow 1=\frac{\tan 9°+\tan 36°}{1-\tan 9° \tan 36°}$$

$\Rightarrow 1-\tan 9° \tan 36° = \tan 9° + \tan 36°$

$\Rightarrow \tan 9° + \tan 36° + \tan 9° \tan 36° = 1$

Illustration 5 Which of the following relations is(are) possible ?

(A) $\sin\theta=\frac{\pi}{2}$ (B) $\tan\theta=2016$

(C) $\cos\theta=\frac{1+t^2}{1-t^2}(t\neq 0,\pm1)$ (D) $\sec\theta=\frac{3}{4}$

Solution (B)

(A) $\sin\theta=\frac{\pi}{2}$ (not possible) as $-1\le\sin\theta\le1\ \forall\ \theta\in R$

(B) $\tan\theta=2016$ (possible) as $\tan\theta$ can take any real value.

(C) $\cos\theta=\frac{1+t^2}{1-t^2}(t\neq 0,\pm1)$ is not possible as

$\frac{1+t^2}{1-t^2}\in(-\infty,-1)\cup(1,\infty)$

(D) $\sec\theta=\frac{3}{4}$

$\Rightarrow \cos\theta=\frac{4}{3}$ (not possible) as $-1\le\cos\theta\le1\ \forall\in R$.

Illustration 6 Let $x=\log_{\text{cosec}\frac{2\pi}{3}}\left(\frac{148}{111}\right)$ and y is the value of tan 780°, then (x –y) tangent of the angle.

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

Solution (C)

$x=2\,;\ y=\sqrt{3}$

$\therefore x-y=2-\sqrt{3} \Rightarrow \tan(x-y)=\frac{\pi}{12}$

Illustration 7 The minimum value of $y = 4\sec^2 x + 9\operatorname{cosec}^2 x$ (wherever defined) is equal to :

(A) 14 (B) 15 (C) 19 (D) 25

Solution (D)

$y = 4\sec^2 x + 9\operatorname{cosec}^2 x$

$y = 4(1+\tan^2 x) + 9(1+\cot^2 x)$

$y = 13 + (2\tan x)^2 + (3\cot x)^2$

$y = 13 + (2\tan x - 3\cot x)^2 + 12$

$y = 25 + (2\tan x - 3\cot x)^2$

$y \geq 25$

Illustration 8 If $x + \frac{1}{x} = 2\cos\theta$, then $x^3 + \frac{1}{x^3}$ is equal to :

(A) $2\sin 2\theta$ (B) $2\sin 3\theta$

(C) $2\cos 2\theta$ (D) $2\cos 3\theta$

Solution (D)

$$x + \frac{1}{x} = 2\cos\theta$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = (2\cos\theta)^3$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 8\cos^3\theta - 6\cos\theta = 2\cos 3\theta$$

Illustration 9 If $k(\operatorname{cosec}15^\circ - \sec 15^\circ) + 3k\left(\tan\frac{\pi}{8} + \cot\frac{\pi}{8}\right)^2 \cdot$

$-(\cos 36^\circ - \sin 18^\circ) = 0$, then k equals

(A) $\frac{1}{24+2\sqrt{2}}$ (B) $\frac{1}{24+2\sqrt{2}}$

(C) $\frac{1}{12+\sqrt{2}}$ (D) $\frac{1}{3(12+\sqrt{2})}$

Solution (B)

$$\cos 36^\circ - \sin 18^\circ = \left(\frac{\sqrt{5}+1}{4}\right) - \left(\frac{\sqrt{5}-1}{4}\right) = \frac{1}{2}$$

$$= \left(\tan\frac{\pi}{8} + \cot\frac{\pi}{8}\right)^2 = \left(\frac{\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}} + \frac{\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}}\right)$$

$$= \left(\frac{1}{\cos\frac{\pi}{8}\cdot\sin\frac{\pi}{8}}\right)^2 = \left(\frac{2}{\cos\frac{\pi}{4}}\right)^2 = \left(2\sqrt{2}\right)^2 = 8$$

$\operatorname{cosec} 15^\circ - \sec 15^\circ$

$$= \frac{2\sqrt{2}}{\sqrt{3}-1} - \frac{2\sqrt{2}}{\sqrt{3}+1} = \frac{2\sqrt{2}(2)}{3-1} = 2\sqrt{2}$$

$k(\operatorname{cosec} 15^\circ - \sec 15^\circ)$

$$+3k\left(\tan\frac{\pi}{8} + \cot\frac{\pi}{8}\right)^2 = (\cos 36^\circ - \sin 18^\circ) = 0$$

$$\Rightarrow k\left(2\sqrt{2}\right) + 3k\cdot(8) - \frac{1}{2} = 0$$

$$\therefore k = \frac{1}{2\left(2\sqrt{2}+24\right)} = \frac{1}{4\left(\sqrt{2}+12\right)}$$

Illustration 10 In triangle ABC, the minimum value of

$\sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2}$ is equal to :

(A) 3 (B) 4 (C) 5 (D) 6

Solution (B)

In ΔABC, $\sum \tan\frac{A}{2}.\tan\frac{B}{2} = 1$

$$\therefore \sum \tan^2\frac{A}{2} \geq \sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$$

$\because$ ($a^2 + b^2 + c^2 - ab - bc - ca \geq 0 \ \forall$ a, b, c$\in$R)

$$\Rightarrow \sum \sec^2\left(\frac{A}{2}\right) = \sum\left(1 + \tan^2\frac{A}{2}\right)$$

$$= 3 + \sum \tan^2\frac{A}{2} \geq 4$$

Illustration 11 If $\frac{\sqrt{1+\sin\frac{79\pi}{8}}}{1+\sin\frac{71\pi}{8}} = \cot\left(\frac{k\pi}{16}\right)$ then the least positive value of k is :

(A) 1 (B) 3 (C) 5 (D) 7

Solution (C)

$$\frac{\sqrt{1-\sin\frac{\pi}{8}}}{\sqrt{1+\sin\frac{\pi}{8}}} = \frac{1-\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}} = \frac{\left(\cos\frac{\pi}{16} - \sin\frac{\pi}{16}\right)^2}{\left(\cos^2\frac{\pi}{16} - \sin^2\frac{\pi}{16}\right)}$$

$$\Rightarrow \frac{\cos\frac{\pi}{16}-\sin\frac{\pi}{16}}{\cos\frac{\pi}{16}+\sin\frac{\pi}{16}} = \frac{1-\tan\frac{\pi}{16}}{1+\tan\frac{\pi}{16}} = \tan\left(\frac{\pi}{4}-\frac{\pi}{16}\right)$$

$$\Rightarrow \tan\frac{3\pi}{16} = \cot\left(\frac{\pi}{2}-\frac{3\pi}{16}\right) = \cot\frac{5\pi}{16}$$

Illustration 12 The maximum value of

$$y = \log_2\frac{(15+3\sin x+4\cos x)}{(15-3\sin x-4\cos x)} \text{ is :}$$

(A) 0 (B) 1 (C) $\log_2 10$ (D) $\log_2 3$

Solution (B)

$$y_{max} = \log_2\left(\frac{15+5}{15-5}\right) = \log_2 2 = 1$$

Illustration 13 If A,B,C are three values lying in $[0, 2\pi]$ for which

$\tan\theta = K$ then $\tan\frac{A}{3}\tan\frac{B}{3}+\tan\frac{B}{3}\tan\frac{C}{3}+\tan\frac{C}{3}\tan\frac{A}{3}$

is equal to :

(A) 3K (B) K (C) –3 (D) –3K

Solution (C)

$\tan\theta = K$

$$\frac{3\tan\frac{\theta}{3}-\tan^3\frac{\theta}{3}}{1-3\tan^2\frac{\theta}{3}} = K$$

$$\tan^3\frac{\theta}{3}-3K\tan^2\frac{\theta}{3}-3\tan\frac{\theta}{3}+K = 0$$

Hence, $\sum\tan\frac{A}{3}\tan\frac{B}{3} = -3$.

Aliter Take $A = 0$, $B = \pi$ and $C = 2\pi$.

Illustration 14 In ΔABC, If $\tan A : \tan B : \tan C = 2 : 3 : 5$ then value of $\sin A \sin B \sin C$ is :

(A) $\frac{5}{14}$ (B) $\frac{\sqrt{3}}{14}$ (C) $\frac{5\sqrt{3}}{7}$ (D) $\frac{5\sqrt{3}}{14}$

Solution (D)

$\tan A = 2k$

$\tan B = 3k$

$\tan C = 5k$

We known if $A + B + C = \pi$ then $\sum \tan A = \pi \tan A$

$10k = 30k^3$.

$$k^2 = \frac{1}{3} \quad \Rightarrow \quad k = \frac{1}{\sqrt{3}}$$

$$\therefore \quad \tan A = \frac{2}{\sqrt{3}};\tan B = \frac{3}{\sqrt{3}};\tan C = \frac{5}{\sqrt{3}}$$

$$\therefore \quad \sin A\sin B\sin C = \frac{2}{\sqrt{7}}\times\frac{\sqrt{3}}{2}\times\frac{5}{\sqrt{28}}$$

$$= \frac{\sqrt{3}\times 5}{2\times 7} = \frac{5\sqrt{3}}{14}$$

Illustration 15 $\sum_{k=1}^{88}(-1)^{k+1}\frac{1}{\sin^2(k+1)^\circ-\sin^2 1^\circ}$ is equal to :

(A) $\tan 2°$ (B) $\cot 2°$

(C) $\frac{\sin 2^\circ}{\cot 2^\circ}$ (D) $\frac{\cot 2^\circ}{\sin 2^\circ}$

Solution (D)

$$\sum_{k=1}^{88}(-1)^{k+1}\frac{1}{\sin^2(k+1)^\circ-\sin^2 1^\circ}$$

$$T_1 = \frac{1}{\sin 1^\circ\sin 3^\circ} = \frac{\sin(3^\circ-1^\circ)}{\sin 2^\circ(\sin 1^\circ\sin 3^\circ)}$$

$$T_2 = \frac{-1}{\sin 2^\circ\sin 4^\circ} = \frac{-\sin(4^\circ-2^\circ)}{\sin 2^\circ(\sin 2^\circ\sin 4^\circ)}$$

$$\sum_{r=1}^{88}T_r = \frac{1}{\sin 2^\circ}[\{(\cot 1^\circ = \cot 3^\circ)+(\cot 3^\circ-\cot 5^\circ)$$
$$+ ...+ (\cot 88^\circ - \cot 89^\circ)\} - \{(\cot 2^\circ - \cot 4^\circ)$$
$$+ (\cot 4^\circ - \cot 6^\circ) + ...+ (\cot 88^\circ - \cot 90^\circ)\}]$$

$$S = \frac{\cot 2^\circ}{\sin 2^\circ}$$

Illustration 16 The continued product $16\prod_{r=1}^{4}\sin\frac{r\pi}{9}$ is equal to

(A) 1 (B) 4 (C) 3 (D) 2

Solution (C)

$E = 16(\sin 20^\circ\cdot\sin 40^\circ\cdot\sin 60^\circ\cdot\sin 80^\circ)$

$$= 16\times\frac{\sqrt{3}}{2}\cos 10^\circ\cdot\cos 50^\circ\cdot\cos 70^\circ$$

$$= \frac{8\sqrt{3}}{2}\cos 30^\circ = 2\sqrt{3}\times\frac{\sqrt{3}}{2} = 3$$

Illustration 17 Let A = sin x cos x. If the expression $\sin^4 x + \cos^4 x$ is expressed as a polynomial in A then it equals.

(A) $\frac{1}{2}+A^2-\frac{1}{2}A^4$ (B) $1-2A^2+A^4$

(C) $\frac{1}{2}-A^2+\frac{1}{2}A^4$ (D) $1+A^2-\frac{1}{2}A^4$

Solution (A)

A = sin x + cos x

$\Rightarrow$ $A^2 = 1 + 2\sin x\cos x$

Now $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin x\cos^2 x = 1 - 2\sin^2 x\cos^2 x$

$$=1-2\left(\frac{A^2-1}{2}\right)=1-\frac{\left(A^2-1\right)^2}{2}$$

$$=\frac{2-\left(A^4-2A^2+1\right)}{2}=\frac{1+2A^2-A^4}{2}$$

$$=\frac{1}{2}+A^2-\frac{1}{2}A^4$$

Illustration 18 If $\cos 2x = \frac{3}{4}$, then the value of $(\sin^4 x + \cos^4 x)$ is equal to :

(A) $\frac{11}{32}$ (B) $\frac{9}{16}$ (C) $\frac{25}{32}$ (D) $\frac{23}{32}$

Solution (C)

$$\cos^4 x + \sin^4 x - 2\sin^2 x\cos^2 x = \frac{9}{16}$$

$$\Rightarrow \cos^4 x+\sin^4 x=\frac{9}{16}+\frac{\sin^2 2x}{2}$$

$$=\frac{9}{16}+\frac{1}{2}\left[1-\frac{9}{16}\right]=\frac{9}{16}+\frac{7}{32}=\frac{18+7}{32}=\frac{25}{32}$$

Illustration 19 Show that

$$\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\cos 9x}{\cos 27x}=\frac{1}{2}[\tan 27x-\tan x]$$

Solution L.H.S,. contains x, 3x, 9x and 27x, whereas R.H.S. contains 27x and x only. So, we will manipulates terms as shown below :

$$\text{R.H.S.} = \frac{1}{2}[\tan 27x-\tan x]$$

$$=\frac{1}{2}[(\tan 27x-\tan 9x)+(\tan 9x-\tan 3x)+(\tan 3x-\tan x)]$$

$$=\frac{1}{2}\left[\left(\frac{\sin 27x}{\cos 27x}-\frac{\sin 9x}{\cos 9x}\right)+\left(\frac{\sin 9x}{\cos 9x}-\frac{\sin 3x}{\cos 3x}\right)+\left(\frac{\sin 3x}{\cos 3x}-\frac{\sin x}{\cos x}\right)\right]$$

$$=\frac{1}{2}\left[\frac{\sin(27x-9x)}{\cos 27x\cos 9x}+\frac{\sin(9x-3x)}{\cos 9x\cos 3x}+\frac{\sin(3x-x)}{\cos 3x\cos x}\right]$$

$$=\frac{1}{2}\left[\frac{\sin 18x}{\cos 27x\cos 9x}+\frac{\sin 6x}{\cos 9x\cos 3x}+\frac{\sin 2x}{\cos 3x\cos x}\right]$$

$$=\frac{1}{2}\left[\frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x}+\frac{2\sin 3x\cos 3x}{\cos 9x\cos 3x}+\frac{2\sin x\cos x}{\cos 3x\cos x}\right]$$

$$=\frac{\sin 9x}{\cos 27x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin x}{\cos 3x}$$

$$=\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}=\text{R.H.S.}$$

Illustration 19 Show that

$$\frac{2\cos 2^n\theta+1}{2\cos\theta+1}=2\left(\cos\theta-1\right)\left(2\cos 2\theta-1\right)$$

$$\left(2\cos 2^2\theta-1\right)\cdots\left(2\cos^{n-1}\theta-1\right)$$

Solution We have to prove $\frac{2\cos 2^n\theta+1}{2\cos\theta+1}$ = (2 cos θ – 1) (2 cos 2θ – 1) (2 cos 2^2θ – 1) ..... (2 cos 2^{n-1} θ – 1)

or 2 cos 2^n θ + 1

= [(2 cos θ + 1) (2 cos θ – 1)] (2 cos 2θ – 1) (2 cos 2^2 θ – 1) ..... (2 cos 2^{n-1} θ – 1)

Now [(2 cos θ + 1) (2 cos θ – 1)] (2 cos 2θ – 1) (2 cos 2^2θ – 1) ..... (2 cos 2^{n-1} θ – 1)

= (4 $\cos^2$ θ – 1) (2 cos 2θ – 1) (2 cos 2^2 θ – 1) ...... (2 cos 2^{n-1} θ – 1)

= (2 cos 2θ + 1) (2 co s 2θ – 1) (2 cos 2^2 θ – 1) ..... (2 cos 2^{n-1} θ –1) [using 2θ = 2 $\cos^2$ θ – 1]

= (4 $\cos^2$ 2θ – 1) (2 cos 2^2 θ – 1) ..... (2 cos 2^{n-1} θ – 1)

= (2 cos 2^2 θ + 1)(2 cos 2^2 θ – 1) ..... (2 cos 2^{n-1} θ – 1)

$= (4\cos^2 2^2\theta - 1)(2\cos 2^3\theta - 1) \ldots\ldots (2\cos 2^{n-1}\theta - 1)$

$\vdots$

$= (2\cos 2^{n-1}\theta + 1)(2\cos 2^{n-1}\theta - 1)$

$= 4\cos^2 2^{n-1}\theta - 1$

$= 2\cos 2^n\theta + 1$

Illustration 20 Prove that

$$\frac{\tan 2^n\theta}{\tan\theta} = (1+\sec 2\theta)(1+\sec 2^2\theta)$$

$$(1+\sec 2^3\theta)\cdots(1+\sec 2^n\theta)$$

Solution We have to prove that

$$\frac{\tan 2^n\theta}{\tan} = (1+\sec 2\theta)(1+\sec^2 2\theta)\ldots\ldots(1+\sec 2^n\theta)$$

or $\quad \tan 2^n\theta = \tan\theta(1+\sec 2\theta)(1+\sec 2^2\theta)\ldots.(1+\sec 2^n\theta)$

Now $\tan\theta(1+\sec 2\theta)(1+\sec 2^2\theta) \ldots\ldots (1+\sec 2^n\theta)$

$$= \tan\theta\left(\frac{1+\cos 2\theta}{\cos 2\theta}\right)(1+\sec 2^2\theta)\ldots.(1+\sec 2^n\theta)$$

$$= \frac{\sin\theta}{\cos\theta}\left(\frac{2\cos^2\theta}{\cos 2\theta}\right)(1+\sec 2^2\theta)\ldots.(1+\sec 2^n\theta)$$

$= (\tan 2\theta)(1+\sec 2^2\theta) \ldots\ldots (1+\sec 2^n\theta)$

$$= (\tan 2\theta)\left(\frac{1+\cos 2^2\theta}{\cos 2^2\theta}\right)(1+\sec 2^3\theta)\ldots\ldots(1+\sec 2^n\theta)$$

$$= (\tan 2\theta)\left(\frac{2\cos^2 2\theta}{\cos 2^2\theta}\right)(1+\sec 2^3\theta)\ldots\ldots(1+\sec 2^n\theta)$$

$= (\tan 2^2\theta)(1+\sec 2^3\theta)\ldots\ldots(1+\sec 2^n\theta)$

$\vdots$

$= \tan 2^{n-1}\theta(1+\sec 2^n\theta)$

$$= \tan 2^{n-1}\theta\left(\frac{1+\cos 2^n\theta}{\cos 2^n\theta}\right)$$

$$= \tan 2^{n-1}\theta\left(\frac{2\cos^2 2^{n-1}\theta}{\cos 2^n\theta}\right)$$

$= \tan 2^n\theta$

Illustration 21 If ABC is a triangle and $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P., then find the minimum value of cot B/2.

Solution $A + B + C = \pi$

or $\quad \frac{A}{2}+\frac{B}{2} = \frac{\pi}{2}-\frac{C}{2}$

or $\quad \cot\left(\frac{A}{2}+\frac{B}{2}\right) = \cot\left(\frac{\pi}{2}-\frac{C}{2}\right)$

or $\quad \dfrac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{A}{2}+\cot\frac{B}{2}} = \tan\frac{C}{2} = \dfrac{1}{\cot\frac{C}{2}}$

or $\quad \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} \qquad .....(i)$

But $\tan\frac{A}{2}, \tan\frac{B}{2}, \cot\frac{C}{2}$ are in H.P.

$\therefore \quad \cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in H.P.

So, $\cot\frac{A}{2}+\cot\frac{C}{2} = 2\cot\frac{B}{2}$

Hence, Eq. (i) becomes

$$\cot\frac{A}{2}+\cot\frac{B}{2}\cot\frac{C}{2} = 2\cot\frac{B}{2} \Rightarrow \cot\frac{A}{2}\cot\frac{C}{2} = 3$$

Thus, G.M. of $\cot\frac{A}{2}$ and $\cot\frac{C}{2}$ is

$$\sqrt{\cot\frac{A}{2}\cot\frac{C}{2}} = \sqrt{3}$$

and A.M. of $\cot\frac{A}{2}$ and $\cot\frac{C}{2}$ is

$$\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{2} = \cot\frac{B}{2}$$

But A.M. $\geq$ G.M. Thus,

$$\cot\frac{B}{2} \geq \sqrt{3}$$

Therefore, the minimum value of cot B/2 is $\sqrt{3}$.

Illustration 22 Find the sum of the series $\operatorname{cosec}\theta + \operatorname{cosec}2\theta + \operatorname{cosec}4\theta + \cdots$ to terms.

Solution $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta}\dfrac{\sin\theta/2}{\sin\theta/2}$

$$= \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\sin\theta \sin\left(\frac{\theta}{2}\right)} = \frac{\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\sin\theta\sin\frac{\theta}{2}}$$

$\therefore$ $\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \cot\theta$

Similarly, $\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta$

Illustration 23 Find the sum of the series $\operatorname{cosec}\theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta + \ldots$ to n terms.

Solution Let $\cot\theta = \cot A + \cot B + \cot C$

or $\cot\theta - \cot A = \cot B + \cot C$

or $\frac{\sin(A-\theta)}{\sin A \sin\theta} = \frac{\sin(B+C)}{\sin B \sin C}$

or $\sin(A-\theta) = \frac{\sin^2 A \sin\theta}{\sin B \sin C}$ .....(i)

Similarly, $\sin(B-\theta) = \frac{\sin^2 A \sin\theta}{\sin B \sin C}$ .....(ii)

and $\sin(C-\theta) = \frac{\sin^2 C \sin\theta}{\sin A \sin B}$ .....(iii)

By multiplying corresponding sides of Eqs. (i), (ii) and (iii), we have

$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$

Illustration 24 If $\tan 6\theta = p/q$, find the value

$\frac{1}{2}(p \operatorname{cosec} 2\theta - q\sec 2\theta)$ in terms of p and q.

Solution Here, we have $\tan 6\theta = p/q$

or $\frac{\sin 6\theta}{\cos 6\theta} = \frac{p}{q}$

or $\frac{p}{\sin 6\theta} = \frac{q}{\cos 6\theta} = \frac{\sqrt{p^2+q^2}}{\sqrt{1}} = \sqrt{p^2+q^2} = k(\text{say})$

Now $y = \frac{1}{2}\left(\frac{1}{\sin 2\theta} - \frac{q}{\cos 2\theta}\right)$

$$= \frac{1}{2}\left(\frac{p\cos 2\theta - q\sin 2\theta}{\sin 2\theta\cos 2\theta}\right) = \frac{1}{2}\left(\frac{p\cos 2\theta - q\sin 2\theta}{\sin 2\theta\cos 2\theta}\right)$$

$$= \left[\frac{2k\sin 6\theta\cos 2\theta - 2k\cos 6\theta\sin 2\theta}{4\sin 2\theta\cos 2\theta}\right]$$

$$= k\frac{\sin(6\theta - 2\theta)}{\sin 4\theta} = k = \sqrt{p^2+q^2}$$

Illustration 25 Eliminate x from equations $\sin(a+x) = 2b$ and $\sin(a-x) = 2c$.

Solution Adding $\sin(a+x) + \sin(a-x) = 2(b+c)$, we get

$2\sin a\cos x + 2(b+c)$

or $\cos x = \frac{b+c}{\sin a}$ .....(i)

Subtracting, we get

$\sin(a+x) - \sin(a-x) = 2(b-c)$

or $2\cos a\sin x = 2(b-c)$

or $\sin x = \frac{b-c}{\cos a}$ .....(ii)

Squaring and adding Eqs. (i) and (ii), we get

$$\frac{(b+c)^2}{\sin^2 a} + \frac{(b-c)^2}{\cos^2 a} = 1$$

EXERCISE – I

SINGLE CORRECT ANSWER TYPE

1. If in a ΔABC, $\cos^3 A + \cos^3 B + \cos^3 C = 3\cos A\cos B\cos C$, then the triangle is :

(A) Isosceles (B) Scalene
(C) equilateral (D) right angled

2. If $\sin\theta = 3\sin(\theta + 2\alpha)$, then the value of $\tan(\theta+\alpha) + 2\tan\alpha$ is :

(A) 0 (B) 2 (C) 4 (D) 1

3. If $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$, then one of the values of $\tan\dfrac{\theta}{2}$ is :

(A) $\tan\dfrac{\theta}{2}\cot\dfrac{\beta}{2}$ (B) $\tan\dfrac{\beta}{2}\cot\dfrac{\alpha}{2}$

(C) $\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}$ (D) None of these

4. If $\alpha + \beta + \gamma = \pi$, $\tan\beta\tan\gamma = 18$ and $\tan\alpha\tan\gamma = 2$ then $\tan^2\gamma =$

(A) 15 (B) 16 (C) 19 (D) 20

5. If $a\cos^2 3\alpha + b\cos^4\alpha = 16\cos^6\alpha + 9\cos^2\alpha$ is identity, then $(a, b) =$

(A) (1, 24) (B) (24, 1) (C) (25, 1) (D) None

6. If $\tan\alpha$, $\tan\beta$ are the roots of the equation $px^2 - qx + r = 0$ where $p \neq 0$, then $\tan(\alpha+\beta)$

(A) $\dfrac{q}{p+r}$ (B) $\dfrac{q}{p-r}$ (C) $\dfrac{p}{q-r}$ (D) $\dfrac{r}{p+q}$

7. If $\tan\left(\dfrac{\pi}{4}+\theta\right) + \tan\left(\dfrac{\pi}{4}-\theta\right) = 3$, then

$\tan^2\left(\dfrac{\pi}{4}+\theta\right) + \tan^2\left(\dfrac{\pi}{4}-\theta\right) = 0$

(A) 6 (B) 4 (C) 7 (D) 5

8. If $\dfrac{\cos x}{a} = \dfrac{\cos(x+\theta)}{b} = \dfrac{\cos(x+2\theta)}{c} = \dfrac{\cos(x+3\theta)}{d}$ then

$\dfrac{a+c}{b+d}$ is equal to

(A) $\dfrac{a}{d}$ (B) $\dfrac{c}{b}$ (C) $\dfrac{b}{c}$ (D) $\dfrac{d}{a}$

9. The value of, $\cot A\cdot\cot(60°+A) + \cot(60°+A)\cdot\cot(120°+A) + \cot(120°+A)\cot A =$

(A) 3 (B) –3 (C) 3/2 (D) 1/2

10. If $\tan x = \dfrac{b}{a}$ then $\left(\dfrac{a+b}{a-b}\right)^{1/2} + \left(\dfrac{a-b}{a+b}\right)^{1/2}$ equals to...... when $x \in \left(0, \dfrac{\pi}{4}\right)$

(A) $\dfrac{2\sin x}{\sqrt{\cos 2x}}$ (B) $\dfrac{2\cos x}{\sqrt{\cos 2x}}$

(C) $\dfrac{2\cos x}{\sqrt{\sin 2x}}$ (D) $\dfrac{2\sin x}{\sqrt{\sin 2x}}$

11. The sum of the series,

$\dfrac{1}{\sin 45^o\sin 46^o} + \dfrac{1}{\sin 47^o\sin 48^o} + \dfrac{1}{\sin 49^o\sin 50^o} + \ldots +$

$\dfrac{1}{\sin 133^o\sin 134^o} = \text{cosec } n^o$ then the integer "n" must be

(A) 1 (B) 45 (C) 90 (D) None

12. $\dfrac{2\sin x}{\sin 3x} + \dfrac{\tan x}{\tan 3x} =$

(A) 1/2 (B) 2 (C) 3/2 (D) 1

13. If $\sin\theta + \sin 2\theta + \sin 3\theta = \sin\alpha$ and $\cos\theta + \cos 2\theta + \cos 3\theta = \cos\alpha$, then θ is equal to :

(A) $\dfrac{\alpha}{2}$ (B) α (C) 2α (D) $\dfrac{\alpha}{6}$

14. The exact value of $\dfrac{96\sin 80^o + \sin 65^o\sin 35^o}{\sin 20^o + \sin 50^o + \sin 110^o}$ is equal to

(A) 12 (B) 24 (C) –12 (D) 48

15. If $\sin\alpha = \dfrac{1}{\sqrt{5}}$ and $\sin\beta = \dfrac{3}{5}$, then all values of $\beta - \alpha$, lies in the interval

(A) $\left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{3\pi}{4}, \pi\right)$ (B) $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{4}\right]$

(C) $\left(\dfrac{3\pi}{4}, \pi\right)$ (D) $\left[\pi, \dfrac{5\pi}{4}\right]$

16. If $\dfrac{\sin\alpha}{\sin\beta} = \dfrac{\sqrt{3}}{2}$ and $\dfrac{\cos\alpha}{\cos\beta} = \dfrac{\sqrt{5}}{2}, 0 < \alpha < \beta < \dfrac{\pi}{2}$, then which of the following is false

(A) $\cot\beta = 1$ (B) $\tan\alpha = \dfrac{\sqrt{3}}{\sqrt{5}}$

(C) $B = \dfrac{\pi}{4}$ (D) $B = \dfrac{\pi}{3}$

17. If sec θ + tan θ = 1, then one root of equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ is :

(A) tan θ (B) sec θ (C) – sec θ (D) sin θ

18. In a ΔPQR, if 3 sin P + 4 cos Q = 6 and 4 sin Q + 3 cos P = 1, then the angle R is :

(A) $\frac{5\pi}{6}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{6}$ (D) $\frac{3\pi}{4}$

19. If x + y = 3 – cos 4θ and x – y = 4 sin 2 θ then :

(A) $x^4 + y^4 = 9$ (B) $\sqrt{x} + \sqrt{y} = 16$

(C) $x^3 + y^3 = 2(x^2 + y^2)$ (D) $\sqrt{x} + \sqrt{y} = 2$

20. The value of the

$$\left(\frac{2(\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + ... + \sin 89^\circ)}{2(\cos 1^\circ + \cos 2^\circ + ... + \cos 44^\circ) + 1}\right)^2$$

expression, equals

(A) 1 (B) 2 (C) $\sqrt{2}$ (D) 3

21. Minimum vertical distance between the graphs of y = 2 + sin x and y = cos x is :

(A) 2 (B) 1 (C) $\sqrt{2}$ (D) $2-\sqrt{2}$

22. The minimum value of the function f(x) = (3 sin x – 4 cos x – 10) (3 sin x + 4 cos x – 10), is

(A) 49 (B) $\frac{195 - 60\sqrt{2}}{2}$

(C) 84 (D) 45

23. If $\frac{\tan\theta}{\tan\theta - \tan 3\theta} = \frac{1}{3}$, then the value of $\frac{\cot\theta}{\cot\theta - \cot 3\theta}$ is :

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) 3 (D) $\frac{-2}{3}$

24. Suppose x and y are real numbers such that tan x + tan y = 42 and cot x + cot y = 49. Find the value of tan(x + y).

(A) 294 (B) 249 (C) 924 (D) None

25. The expression $\frac{1 + \sin 2\alpha}{\cos(2\alpha - 2\pi)\cdot\tan\left(\alpha - \frac{3\pi}{4}\right)}$

$$-\frac{1}{4}\sin 2\alpha\left[\cot\frac{\alpha}{2} + \cot\left(\frac{3\pi}{2} + \frac{\alpha}{2}\right)\right]$$

when simplified reduces to :

(A) 1 (B) 0

(C) $\sin^2(a/2)$ (D) $\sin^2\alpha$

26. The expression,

$$\frac{\tan\left(x - \frac{\pi}{2}\right)\cdot\cos\left(\frac{3\pi}{2} + x\right) - \sin^3\left(\frac{7\pi}{2} - x\right)}{\cos\left(x - \frac{\pi}{2}\right)\cdot\tan\left(\frac{3\pi}{2} + x\right)}$$ simplifies to :

(A) $(1 + \cos^2)$ (B) $\sin^2 x$

(C) $-(1 + \cos^2 x)$ (D) $\cos^2 x$

27. The expression S = sec 11° · sec 19° – 2 cot 71° reduces to:

(A) 2 cot 11° (B) tan 19°

(C) 2 tan 11° (D) $\frac{1}{2}\tan 19^\circ$

28. If $\cos^2\theta = \frac{1}{3}(a^2 - 1)$ and $\tan^2 = \frac{\theta}{2} = \tan^{2/3}\alpha$, then $\cos^{2/3}\alpha + \sin^{2/3}\alpha =$

(A) $2a^{2/3}$ (B) $\left(\frac{2}{a}\right)^{2/3}$ (C) $\left(\frac{2}{a}\right)^{1/3}$ (D) $2a^{1/3}$

29. If $0 \le x \le \frac{\pi}{3}$ then Range of

$f(x) = \sec\left(\frac{\pi}{6} - x\right) + \sec\left(\frac{\pi}{6} + x\right)$ **is :**

(A) $\left(\frac{4}{\sqrt{3}}, \infty\right)$ (B) $\left[\frac{4}{\sqrt{3}}, \infty\right)$

(C) $\left(0, \frac{4}{\sqrt{3}}\right]$ (D) $\left(0, \frac{4}{\sqrt{3}}\right)$

30. Let $a_1 = \left(\tan\frac{\pi}{8}\right)^{\tan\frac{\pi}{8}}, a_4 = \left(\cot\frac{\pi}{8}\right)^{\cot\frac{\pi}{8}}$. Then which one of the following statements is true about relative sizes of a_1, a_2, a_3, a_4?

(A) $a_4 > a_3 > a_2 > a_1$ (B) $a_3 > a_4 > a_2 > a_1$

(C) $a_4 > a_3 > a_1 > a_2$ (D) $a_3 > a_1 > a_2 > a_4$

31. In ΔABC, the minimum value of $\frac{\sum \cot^2\frac{A}{2}\cdot\cot^2\frac{B}{2}}{\prod \cot^2\frac{A}{2}}$ is :

(A) 1 (B) 2 (C) 3 (D) Non existent

32. Let x = (sin α) $\log_b$ sin α, y = (cos α) $\log_b$ cos α and z = (sin α) $\log_b$ cos α, w = (cos α)$\log_b$ sin α where 0 < b < 1 and $0 < \alpha < \frac{\pi}{4}$. Then

(A) x < z < w < y (B) y < z < x < w

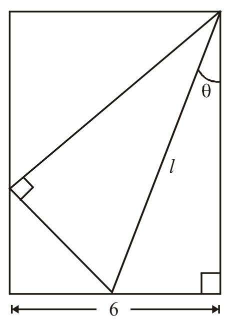
(C) z < x < 0 < y (D) x < y < z < w

33. If w = $\log_{\sin 1}$ cos 1, x = $\log_{\sin 1}$ tan 1, y = $\log_{\cos 1}$ sin 1 and z = $\log_{\cos 1}$ tan 1 then

(A) w < y < x < z (B) y < z < w < z

(C) x < z < y < w (D) z < x < w < y

34. One side of a rectangular piece of paper is 6 cm, the adjacent sides being longer than 6 cms. One corner of the paper is folded so that it sets on the opposite longer side. If the length of the crease is *l* cms and it makes an angle θ with the long side as shown, then *l* is



(A) $\frac{3}{\sin\theta\cos^2\theta}$ (B) $\frac{6}{\sin^2\theta\cos\theta}$

(C) $\frac{3}{\sin\theta\cos\theta}$ (D) $\frac{3}{\sin\theta}$

35. If the roots of the equations $x^3 - px^2 - r = 0$ are tan α, tan β tan γ then the value of $\sec^2\alpha \cdot \sec^2\beta \cdot \sec^2\gamma$ is

(A) $p^2 + r^2 + 2rp + 1$ (B) $p^2 + r^2 - 2rp + 1$

(C) $p^2 - r^2 - 2rp + 1$ (D) None

36. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$, then x + y + z is equal to :

(A) 1 (B) 0 (C) –1 (D) None of these

37. The roots of the equation $4x^2 - 2\sqrt{5}x + 1 = 0$ are :

(A) sin 36°, sin 18° (B) sin 18°, cos 36°

(C) sin 36°, cos 18° (D) cos18°, cos 36°

38. If f(θ) = 5 cosθ + 3 cos $\left(\theta + \frac{\pi}{3}\right) + 3$, then range of f(θ) is:

(A) [–5, 11] (B) [–3, 9] (C) [–2, 10] (D) [–4, 10]

39. If α, β, γ, δ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the value of $4\sin\frac{\alpha}{2} + 3\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} + \sin\frac{\delta}{2}$ is equal to :

(A) $2\sqrt{1-k}$ (B) $2\sqrt{1+k}$

(C) $\frac{\sqrt{1+k}}{2}$ (D) None of these

40. If θ = π/4n, then the value of tanθ tan2θ ... tan (2n –2)θ tan (2n –1)θ is :

(A) –1 (B) 1 (C) 0 (D) 2

41. If A = sin 45° + cos 45° and B = sin 44° + cos 44°, then

(A) A > B (B) A < B

(C) A = B (D) None of these

42. If $\cos\theta_1 = 2\cos\theta_2$, then $\tan\frac{\theta_1 - \theta_2}{2}\tan\frac{\theta_1 + \theta_2}{2}$ is equal to

(A) $\frac{1}{3}$ (B) $-\frac{1}{3}$ (C) 1 (D) –1

43. In triangle ABC, If sinA cosA = $\frac{1}{4}$ and 3 tan A = tan B, then $\cot^2$ A is equal to :

(A) 2 (B) 3 (C) 4 (D) 1

44. tan 100° + tan 125° + tan 100° tan 125° is equal to :

(A) 0 (B) 1/2 (C) –1 (D) 1

45. If tan α is equal to the integral solution of the inequality $4x^2 - 16x + 15 < 0$ and cos β is equal to the slope of the bisector of the first quadrant, then sin(α + β) sin (α – β) is equal to :

(A) $\frac{3}{5}$ (B) $\frac{3}{5}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{4}{5}$

46. If in triangle ABC, sin A cos B = 1/4 and 3 tan A = tan B, then the triangle is :

(A) right angled (B) equilateral

(C) isosceles (D) None of these

47. Let f(θ) $\frac{\cot\theta}{1 + \cot\theta}$ and $\alpha + \beta = \frac{5\pi}{4}$, then the value f(α) f(β) is :

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 2 (D) None of these

48. If y = (1+ tan A) (1 – tan B), where $A - B = \frac{\pi}{4}$, then $(y+1)^{y+1}$, is equal to :

(A) 9 (B) 4 (C) 27 (D) 81

49. If $x_1, x_2, x_3, \ldots, x_n$ are in A.P. whose common difference is α, then the value of sin a (sec x_1 sec_2 + sec_2 sec_3 + ...+ sec x_{n-1} sec x_n) is :

(A) $\frac{\sin(n-1)\alpha}{\cos x_i \cos x_n}$ (B) $\frac{\sin n\alpha}{\cos x_i \cos x_n}$

(C) sin(n –1) α cos x_1 cos x_n (D) sin n α cos x_1 cos x_n

50. If $\tan\frac{\pi}{9}, x$ and $\tan\frac{5\pi}{18}$ are in A.P. and $\tan\frac{\pi}{9}$, y and $\tan\frac{7\pi}{18}$ are also in A.P. then

(A) 2x = y (B) x > 2

(C) x = y (D) None of these

51. Let x = sin 1°, then the value of the expression,

$$\frac{1}{\cos 0^\circ \cdot \cos 1^\circ} + \frac{1}{\cos 1^\circ . \cos 2^\circ} + \frac{1}{\cos 2^\circ . \cos 3^\circ} + \ldots + \frac{1}{\cos 44^\circ . \cos 45^\circ}$$ is equal to :

(A) x (B) 1/x (C) $\sqrt{2}/x$ (D) $x/\sqrt{2}$

52. If x, y, z are in A.P., then $\frac{\sin x - \sin z}{\cos z - \cos x}$ is equal to :

(A) tan 3θ (B) cot 3θ

(C) sin y (D) cos y

53. If x_1 and x_2 are two distinct roots of the equation, a cos x + b sin x = c, then $\tan\frac{x_1 + x_2}{2}$ is equal to :

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{c}{a}$ (D) $\frac{a}{c}$

54. Give that $\left(1+\sqrt{1+x}\right)\tan y = 1+\sqrt{1-x}$. Then sin 4y is equal to :

(A) 4x (B) 2x (C) x (D) None of these

55. If $\tan\theta = \sqrt{n}$, where n ∈ N, ≥ 2, then sec 2θ is always

(A) A rational number (B) an irrational number

(C) A positive integer (D) A negative integer

56. $\cos^3 x \sin 2x = \sum_{x=0}^{n} a_r \sin(r x) \forall x \in R$, then

(A) n = 5, a_1 = 1/2 (B) n = 5, a_1 = 1/4

(C) n = 5, a_1 = 1/8 (D) n = 5, a_2 = 1/4

57. If A + B + C = 3π/2, then cos 2A + cos 2B + cos 2C is equal to :

(A) 1 – 4 cos A cos B cos C

(B) 4 sin A sin B sin C

(C) 1 + 2 cos A cos B cos C

(D) 1–4 sin A sin B sinC

58. $\tan^6\frac{\pi}{9} - 33\tan^4\frac{\pi}{9} + 27\tan^2\frac{\pi}{9}$ is equal to :

(A) 0 (B) $\sqrt{3}$ (C) 3 (D) 9

59. If cos x cos y – cos(x + y) = $\frac{3}{2}$, then

(A) x + y = 0 (B) x = 2y

(C) x = y (D) 2x = y

60. Given that a, b, c are the sides of a triangle ABC which is right angled at C, then the minimum value of $\left(\frac{c}{a}+\frac{c}{b}\right)^2$ is :

(A) 0 (B) 4 (C) 6 (D) 8

EXERCISE –II

MULTIPLE CORRECT ANSWER TYPE

1. If $\frac{\sin^4 x}{5} + \frac{\cos^4 x}{4} = \frac{1}{9}$, then which of the following is/are true ?

(A) $\cot^2 x = \frac{4}{5}$ (B) $\tan^2 x = \frac{4}{5}$

(C) $\frac{64}{\cos^6 x} + \frac{125}{\sin^6 x} = 1458$ (D) $\frac{125}{\cos^6 x} + \frac{64}{\sin^6 x} = 1458$

2. If cos x + cos y = a, cos 2x + cos 2y = b, cos 3x + cos 3y = c, then

(A) $\cos^2 x + \cos^2 y = 1 + \frac{b}{2}$

(B) $\cos x \cdot \cos y = \frac{a^2}{2} - \left(\frac{b+2}{4}\right)$

(C) $2a^3 + c = 3a(1 + b)$

(D) a + b + c = 3 abc

3. If $\frac{1+\cos 2x}{\sin 2x}+3\left(1+(\tan x)\tan\frac{x}{2}\right)\sin x=4$, then the value of tan x can be equal to

(A) 1 (B) $\frac{1}{2}$ (C) 3 (D) $\frac{1}{3}$

4. If $\sin\alpha=\frac{3}{5},\frac{\pi}{2}<\alpha<\pi$ and $\cos\beta=\frac{-5}{13},\pi<\beta<\frac{3\pi}{2}$, then the correct statements is (are)

(A) $\tan(\alpha-\beta)=\frac{63}{16}$ (B) $\tan(\alpha+\beta)=\frac{33}{56}$

(C) $\sin 2\alpha=\frac{-24}{25}$ (D) $\cos 2\beta=\frac{-119}{169}$

5. Which of the following identities wherever defined hold(s) good ?

(A) $\cos 3\theta = 4\cos\theta - 3\cos^3\theta$

(B) $\tan\left(\frac{\pi}{4}-\theta\right)=\frac{1-\sin 2\theta}{\cos 2\theta}$

(C) $\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(D) $\sin 3\theta = \sin\theta\cdot\sin(60°-\theta)\cdot\sin(60°+\theta)$

6. Which of the following when simplified reduces to unity ?

(A) $\sin(-420°)\cdot\cos(390°)+\cos(-660°)\sin(330°)$

(B) $\cos^2\left(\frac{3\pi}{8}+\theta\right)-\sin^2\left(\frac{\pi}{8}-\theta\right)$

(C) $\sin(290°-\theta)\cdot\cos(160°+\theta)+\cos(290°-\theta)\cdot\sin(160°+\theta)$

(D) $\cos 200°\cdot\cos 160°-\sin 200°\cdot\sin 160°$

7. If x_1 and x_2 are two real solutions of the equation $(x)^{\ln x^2}=e^{18}$ then the product (x_1x_2) equals.

(A) $\frac{(\cot^2 5^o)(\cos^2 5^o)}{\cot^2 5^o-\cos^2 5^o}$

(B) $\frac{(\tan^2 10^o)(\sin^2 10^o)}{\tan^2 10^o-\sin^2 10^o}$

(C) $\sec 0+\sec\frac{\pi}{7}+\sec\frac{2\pi}{7}+\sec\frac{3\pi}{7}+\sec\frac{4\pi}{7}+\sec\frac{5\pi}{7}+\sec\frac{6\pi}{7}$

(D) $\sqrt{1-\sin^2 110^o}\cdot\sec 110^o$

8. Which of the following is(are) **INCORRECT ?**

(A) cos (–1) < cos 2 (B) cos 2 > cos 3

(C) sin 3 > sin 2 (D) sin 1 < cos 1

9. Which of the following quantities is(are) rational ?

(A) $\sin^6\left(\frac{\pi}{8}\right)+\cos^6\left(\frac{\pi}{8}\right)$

(B) $\frac{3\tan 10^o-\tan^3 10^o}{1-3\tan^2 10^o}$

(C) cosec 10° · cosec 50° · cosec 70°

(D) $\sin\left(\frac{7\pi}{10}\right)\cdot\cos\left(\frac{3\pi}{5}\right)$

10. The value of $\sum_{k=1}^{5}\frac{1}{\sin(k+1)\cdot\sin(k+2)}$ is :

(A) positive (B) negative

(C) $\frac{\cot 2-\cot 7}{\sin 1}$ (D) $\frac{\tan 2-\tan 7}{\sin 1}$

11. If $f_n(x)=\prod_{s=0}^{n-1}\left(2\cos\left(2^s x\right)-1\right), n\geq 1$. Then which of the following is (are) incorrect ?

(A) $f_5\left(\frac{2\pi}{33}\right)=-1$ (B) $f_5\left(\frac{6\pi}{255}\right)=1$

(C) $f_5\left(\frac{2\pi}{31}\right)=1$ (D) $f_5\left(\frac{4\pi}{127}\right)=1$

12. Let $a=5^{\log_5(\sin x)}$ and $b=7^{\log_7(\cos x)}$ then $\frac{a}{b}$ can be equal to

(A) tan 2° (B) tan 4° (C) tan 5° (D) tan 1°

13. In ΔABC if sin A sin(B – C) = sin C sin (A – B), then (where A ≠ B ≠ C).

(A) tan A, tan B, tan C are in arithmetic progression

(B) cot A, cot B, cot C are in arithmetic progression

(C) cos 2A, cos 2B,cos 2C are in arithmetic progression

(D) sin 2A, sin 2B, sin 2C are in arithmetic progression

14. If $y=\dfrac{\sqrt{1-\sin 4A}+1}{\sqrt{1+\sin 4A}-1}$ then one of the values of y is :

(A) $-\tan A$ (B) $\cot A$

(C) $\tan\left(\dfrac{\pi}{4}+A\right)$ (D) $-\cot\left(\dfrac{\pi}{4}+A\right)$

15. If $\dfrac{\tan 3A}{\tan A}=k,(k\neq 1)$, then

(A) $\dfrac{\cos A}{\cos 3A}=\dfrac{k^2-1}{2k}$ (B) $\dfrac{\sin 3A}{\sin A}=\dfrac{2k}{k-1}$

(C) $k<\dfrac{1}{3}$ (D) $k>3$

16. If cos x – sin α cot β sin x = cos α, then $\tan\dfrac{x}{2}$ is equal to:

(A) $\cot\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$ (B) $-\cot\dfrac{\beta}{2}\tan\dfrac{\alpha}{2}$

(C) $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$ (D) $\cot\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$

17. If (a – b) sin (θ + ϕ) = (a + b) sin (θ + ϕ) and a $\tan\dfrac{\theta}{2}-b\tan\dfrac{\phi}{2}=c$, then

(A) $b\tan\phi=a\tan\theta$ (B) $a\tan\phi=b\tan\theta$

(C) $\sin\phi=\dfrac{2bc}{a^2-b^2-c^2}$ (D) $\sin\theta=\dfrac{2ac}{a^2-b^2+c^2}$

18. Consider two inequalities $(\sin x)^{\sin x}<(\cos x)^{\cos x}$ and $(\sin x)^{\sin x}>(\cos x)^{\cos x}$ in $\left(0,\dfrac{\pi}{2}\right)$.

(A) First is true second is false.

(B) First is false second is true

(C) Both are false

(D) The second is true at $x=\dfrac{\pi}{3}$

19. If $7\cos x-24\sin x=\lambda\cos(x-\alpha)$, $0<\alpha<\dfrac{\pi}{2}$ be true for all $x\in R$ then (or) p = 10, q = 8

(A) $\lambda=25$ (B) $\alpha=\sin^{-1}\dfrac{24}{25}$

(C) $\lambda=-25$ (D) $\alpha=\cos^{-1}\dfrac{7}{25}$

20. If $A+B=\dfrac{\pi}{3}$ and cos A + cos B = 1, then which of the following is false ?

(A) $\sin(A-B)=\dfrac{2\sqrt{2}}{3}$ (B) $|\cos A-\cos B|=\sqrt{\dfrac{2}{3}}$

(C) $\cos(A-B)=-\dfrac{1}{3}$ (B) $|\cos A-\cos B|=\dfrac{1}{2\sqrt{3}}$

21. Let $y=\dfrac{\sum_{r=1}^{7}\cos rx}{\sum_{r=1}^{7}\sin rx}$ then which of the following hold good ?

(A) The value of y when $x=\dfrac{\pi}{8}$ is not defined.

(B) The value of y when $x=\dfrac{\pi}{16}$ is 1.

(C) The value of y when $x=\dfrac{\pi}{32}$ is $\sqrt{2}-1$.

(D) The value of y when $x=\dfrac{\pi}{48}$ is $2+\sqrt{3}$.

22. If sin θ + sin ϕ = a and cos θ + cos ϕ = b, then

(A) $\cos\left(\dfrac{\theta-\phi}{2}\right)=\pm\dfrac{1}{2}\sqrt{(a^2+b^2)}$

(B) $\cos\left(\dfrac{\theta-\phi}{2}\right)=\pm\dfrac{1}{2}\sqrt{(a^2-b^2)}$

(C) $\tan\left(\dfrac{\theta-\phi}{2}\right)=\pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$

(D) $\cos(\theta-\phi)=\dfrac{a^2+b^2-2}{2}$

23. Which of the following quantities are rational ?

(A) $\sin\left(\dfrac{11\pi}{12}\right)\sin\left(\dfrac{5\pi}{12}\right)$

(B) $\operatorname{cosec}\left(\dfrac{9\pi}{10}\right)\sec\left(\dfrac{4\pi}{5}\right)$

(C) $\sin^4\left(\dfrac{\pi}{8}\right)+\cos^4\left(\dfrac{\pi}{8}\right)$

(D) $\left(1+\cos\dfrac{2\pi}{9}\right)\left(1+\cos\dfrac{4\pi}{9}\right)\left(1+\cos\dfrac{8\pi}{9}\right)$

24. Which of the following inequalities hold true in any triangle ABC ?

(A) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$

(B) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\le\frac{3\sqrt{3}}{8}$

(C) $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}<\frac{3}{4}$

(D) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\le\frac{9}{4}$

25. For $\alpha = \pi/7$, which of the following hold (s) good ?

(A) $\tan\alpha\tan2\alpha\tan3\alpha = \tan3\alpha - \tan2\alpha - \tan\alpha$

(B) $\operatorname{cosec}\alpha = \operatorname{cosec}2\alpha + \operatorname{cosec}4\alpha$

(C) $\cos\alpha - \cos2\alpha + \cos3\alpha = 1/2$

(D) $8\cos\alpha\cos2\alpha\cos4\alpha = 1$

26. Which of the following identities, wherever defined, hold(s) good ?

(A) $\cot\alpha - \tan\alpha = 2\cot2\alpha$

(B) $\tan(45° + \alpha) - \tan(45° - \alpha) = 2\operatorname{cosec}2\alpha$

(C) $\tan(45° + \alpha) + \tan(45° - \alpha) = 2\sec2\alpha$

(D) $\tan\alpha + \cot\alpha = 2\tan2\alpha$

27. The expression $(\tan^4x + 2\tan^2x + 1)\cos^2x$ when $x = \pi/12$ can be equal to

(A) $4\left(2-\sqrt{3}\right)$ (B) $4\left(\sqrt{2}+1\right)$

(C) $16\cos^2\pi/12$ (D) $16\sin^2\pi/12$

28. Let α, β and γ be some angles in the first quadrant satisfying $\tan(\alpha + \beta) = 15/8$ and $\operatorname{cosec}\gamma = 17/8$, then which of the following hold (s) good ?

(A) $\alpha + \beta + \gamma = \pi$

(B) $\cot\alpha\cot\beta\cot\gamma = \cot\alpha + \cot\beta + \cot\gamma$

(C) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

(D) $\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1$

29. The equation $x^3-\frac{3}{4}x=-\frac{\sqrt{3}}{8}$ is satisfied by

(A) $x=\cos\left(\frac{5\pi}{18}\right)$ (B) $x=\cos\left(\frac{7\pi}{18}\right)$

(C) $x=\cos\left(\frac{23\pi}{18}\right)$ (D) $x=\cos\left(\frac{17\pi}{18}\right)$

EXERCISE –III

PART - I ASSERTION REASON

1. Statement - 1 : $\frac{\cos36^o-\cos72^o}{\cos36^o\cos72^o}=2$

Statement - 2 : $\sin15^o=\frac{\sqrt{6}-\sqrt{2}}{4}$

2. Statement - 1 : $(1 + \cot^o)(1 + \cot2^o).....(1 + \cot45^o)$ $= 2^{23}\cot1^o\cot2^o ..... \cot44^o\cot45^o$

Statement - 2 : $(1 - \cot1^o)(1 - \cot2^o).......(1 - \cot45^o) = 2^{23}$

3. Statement - 1 : If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta = y\cos\theta$, then $x^2 + y^2 = 1$.

Statement - 2 : $4\sin^3\theta = 3\sin\theta - \sin3\theta$ and $4\cos^3\theta = 3\cos\theta + \cos3\theta$

4. Statement - 1 : $\tan5\theta - \tan3\theta - \tan2\theta = \tan5\theta\tan3\theta\tan2\theta$.

Statement - 2 : $x = y + z \Rightarrow \tan x - \tan y - \tan z = \tan x\tan y\tan z$.

5. Statement - 1 : The maximum value of $\sin\theta + \cos\theta$ is 2

Statement - 2 : The maximum value of $\sin\theta$ is 1 and that of $\cos\theta$ is also 1.

6. Statement - 1 : $\sec^2\theta=\frac{4xy}{(x+y)^2}$ is possible for all real values of x and y only when x = y

Statement - 2 : $\sec^2\theta \ge 1$.

7. Statement - 1 : $\sin\theta = x+\frac{1}{x}$ is impossible if $x \in R \sim \{0\}$.

Statement - 2 : $x+\frac{1}{x}\in(-\infty,-2]\cup[2,\infty)$

8. Statement - 1 : If A > 0, B > 0 and $A + B = \frac{\pi}{3}$ then the maximum value of tan A tan B is $\frac{1}{3}$.

Statement - 2 : If $a_1 + a_2 + a_3 + ... + a_n = k$ (constant) then value $a_1 a_2 a_3 ...a_n$ is greatest when $a_1 = a_2 = a_3 = ... = a_n$

9. Statement - 1 : sin 3 < sin 1 < sin 2 is true

Statement - 2 : sin x is positive in first and second quadrants.

10. Statement - 1 : Maximum value of $\begin{vmatrix}1 & 3\cos\theta & 1\\ \sin\theta & 1 & 3\cos\theta\\ 1 & \sin\theta & 1\end{vmatrix}$ is 10

Statement - 2 : Maximum value of $(a \cos \theta + b \sin \theta)^2$ is $a^2 + b^2$.

11. **Statement - 1** : $\cos 1 < \cos 7$.

Statement - 2 : $1 < 7$.

12. **Statement - 1** : If $f(\theta) = (\sin\theta + \text{cosec}\theta)^2 + (\cos \theta + \sec \theta)^2$, then the minimum value of $f(\theta)$ is 9.

Statement - 2 : Maximum value of $\sin^2\theta \cdot \cos^2\theta$ is $\frac{1}{4}$.

13. **Statement - 1** : The minimum value of the expression $\sin \alpha + \sin \beta + \sin \gamma$ is negative, where α, β, γ are real numbers such that $\alpha + \beta + \gamma = \pi$.

Statement - 2 : All sin a, sin b, sin g are positive.

14. **Statement - 1** : If $x + y + z = xyz$, then at most one of the numbers can be negative.

Statement - 2 : In a triangle ABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ and there can be at most one obtuse angle in a triangle.

15. **Statement - 1** : The maximum value of $\sin \sqrt{2}x + \sin ax$ cannot be 2 (a is positive rational number).

Statement - 2 : $\frac{\sqrt{2}}{a}$ is irrational.

16. **Statement - 1** : $\sin \pi/18$ is a root of $8x^3 - 6x + 1 = 0$.

Statement - 2 : For any $\theta \in R$, $\sin^3\theta = 3 \sin \theta - 4 \sin^3\theta$.

17. **Statement - 1** : In a triangle, the least value of the sum of cosines of its angles is unity.

Statement - 2 : $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, if A, B, C are the angles of a triangle.

18. If $A + B + C = \pi$, then

Statement - 1 : $\cos^2 A + \cos^2 B + \cos^2 C$ has its minimum value $\frac{3}{4}$.

Statement - 2 : Maximum value of $\cos A \cos B \cos C$ is $\frac{1}{8}$.

PART - II COMPREHENSION TYPE

Comprehension # 1

Let $P_n(x) = \frac{1}{n}\left(\sin^n x + \cos^n x\right) \forall n \in N$ and

$Q(m) = \cos(63°) + (\cos 57°)^m + (\cos 63°)^{m-1} (\cos 57°)^{m-1}$

$\forall$ m $\in$ N. Also given $\log_{10}2 = 0.3010$; $\log_{10}3 = 0.4771$.

1. The value of $\log_{1.\bar{3}}(Q(2))$ is equal to :

(A) 0 (B) 2 (C) 1 (D) –1

2. The value of $(12P_4(x) - P_6(x)$ at $x = \frac{\pi}{10}$ equals

(A) 1 (B) 3 (C) 6 (D) 1/12

3. Number of zeroes in $\left(\frac{8}{9}P_2(x)\right)^{100}$ after decimal before a significant figures starts, is :

(A) 30 (B) 31 (C) 35 (D) 36

Comprehension # 2

Let $a = \cos 10°$, $b = \cos 50°$ and $c = \cos 70°$.

1. The value of $(a^2 + b^2 + c^2)$ is equal to :

(A) 2 (B) $\frac{3}{2}$ (C) $\frac{4}{3}$ (D) 1

2. The value of $(b + c)$ is equal to :

(A) a (B) $\frac{a}{2}$ (C) 2a (D) $\frac{2a}{3}$

3. The value of $4(a^2 - bc)$ is equal to :

(A) $\frac{8}{3}$ (B) $\frac{5}{2}$ (C) 2 (D) 3

Comprehension # 3

Consider $\cos^2\theta - \sin^2\theta = \sqrt{\frac{5+\sqrt{5}}{8}}, \theta \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$.

1. Number of values of θ satisfying the given equation is :

(A) 2 (B) 3 (C) 4 (D) 5

2. The difference between greatest and least value of θ is :

(A) $\frac{\pi}{5}$ (B) $\frac{\pi}{15}$ (C) $\frac{\pi}{20}$ (D) $\frac{\pi}{10}$

Comprehension # 4

Let $\sin \alpha + \sin \beta = \frac{\sqrt{6}}{3}$ and $\cos \alpha + \cos \beta = = \frac{\sqrt{3}}{3}$.

1. The value of $\tan\left(\frac{\alpha+\beta}{2}\right)$ is equal to :

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $\frac{1}{\sqrt{3}}$

2. The value of $100 \cos^2\left(\frac{\alpha-\beta}{2}\right)$ is equal to :

(A) 5 (B) 25 (C) 50 (D) 75

Comprehension # 5

Consider the polynomial P(x) = (x – tan 10°) (x – tan 50°) (x – tan 70°)

1. Constant term in P(x) has the value

(A) $\frac{-1}{\sqrt{3}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $-\sqrt{3}$ (D) $\sqrt{3}$

2. If a, b and c are coefficient of x^2, coefficients of x and constant term in P(x) respectively, then the value of $\frac{-a+c}{1-b}$ is equal to :

(A) tan 130° (B) tan 120°
(C) tan(–130°) (D) tan 150°

Comprehension # 6

For n∈N let $f(n) = 8(\cos^n\theta - \sin^n\theta)$. $g(n) = 8(\cos^n\theta + \sin^n\theta)$

1. If f(6) = a cos(2θ) + b cos (6θ), ∀ θ ∈R then the value of a + b is equal to

(A) 8 (B) 7 (C) 6 (D) 5

2. The value of $\left(\frac{g(6)}{32}+\frac{3}{16}\sin^2 2\theta\right)$ is equal to :

(A) 0 (B) $\frac{1}{4}$ (C) $\frac{-1}{4}$ (D) –2

Comprehension # 7

Let N be the number of solution of equation,
$(\log_{\tan x}\sin x)(\log_{\cos x} x)(\log_x \tan x) = 1$ in [0, 2π].
M be the sum of integral values of equation,

$2\log_8(2x) + \log_8(x^2 - 2x + 1) = \frac{4}{3}$.

P be the value of y in equation.
$(8\cos^2 9° - 6)(12\cos^2 27° - 9) = 6\tan y°$, y ∈(0,10)

1. The value of $M^2 + N^2$ is equal to :

(A) 4 (B) 5 (C) 6 (D) 8

2. The value of P + M is equal to

(A) 8 (B) 9 (C) 10 (D) 11

Comprehension # 8

Given that $\frac{\cos^4\alpha}{\cos^2\beta}+\frac{\sin^4\alpha}{\sin^2\beta}=1$

1. $\frac{\sin^2\alpha}{\sin^2\beta}+\frac{\sin^2\beta}{\sin^2\alpha}=$

(A) 1 (B) 2 (C) 3 (D) 4

2. $\frac{\cos^2\alpha}{\cos^2\beta}+\frac{\cos^2\beta}{\cos^2\alpha}=$

(A) 1 (B) 2 (C) 3 (D) 4

3. $\frac{\cos^4\beta}{\cos^2\alpha}+\frac{\sin^4\beta}{\sin^2\alpha}=$

(A) 1 (B) 2 (C) 3 (D) 4

Comprehension # 9

Consider the polynomial P(x) = (x – cos 36°) (x – cos 84°) (x – cos 156°)

1. The coefficient of x^2 is :

(A) 0 (B) 1 (C) $-\frac{1}{2}$ (D) $\frac{\sqrt{5}-1}{2}$

2. The coefficient of x is :

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $-\frac{3}{4}$ (D) Zero

3. The absolute term in P(x) has the value equal to :

(A) $\frac{\sqrt{5}-1}{4}$ (B) $\frac{\sqrt{5}-1}{16}$ (C) $\frac{\sqrt{5}+1}{16}$ (D) $\frac{1}{16}$

PART - III MATCH THE COLUMN

1.

Column–I	Column–II
(A) If cosec θ + sin θ = 3, then $\log_5(\operatorname{cosec}^5\theta + \sin^5\theta)$ is greater than	(P) 1
(B) The value of $\cos^2\frac{\pi}{18}+\cos^2\frac{\pi}{9}+\cos^2\frac{7\pi}{18}+\cos^2\frac{4\pi}{18}$, is coprime with	(Q) 2
(C) If $5^{(\log_5 7)^{2x}} = 7^{(\log_7 5)^x}$ then the value of x is smaller than	(R) 3
(D) If $\log(xy^3) = 1$, $\log(x^2y) = 1$, then value of $\log(xy)^5$ is twin prime with	(T) 5

2.

Column- I	Column-II
(A) If $\cos^2 x + 5\cos x = 2\sin^2 x$, then the value of sec x is greater than	(P) 1
(B) The value of $(0.2)\left(1+9^{\log_3 8}\right)^{\log_{65} 5}$ is coprime with	(Q) 2

(C) If the expression, (R) 3

$\cos\left(x-\frac{3\pi}{2}\right)+\sin\left(\frac{3\pi}{2}+x\right)+\sin(32\pi+x)$

$-18\cos(19\pi-x)+\cos(56\pi+x)-9\sin(x+17\pi)$

is expressed in the form of a sin x + b cos x,

then the value of b/a equals. (S) 4

3. **Column- I** **Column-II**

(A) If tan 50° – tan 40° = k tan 10° then the value of k is (P) 1

(B) Number of expression (s) (given below), which simplifies to $(\sec^2\theta - \text{cosec}^2\theta)$, is twin prime with (Q) 2

(i) $\sec^2\theta + \text{cosec}^2\theta$

(ii) $(\tan\theta+\cot\theta)^2$

(iii) $\frac{\tan^2\theta+1}{1-\cos^2\theta}$

(iv) $\text{cosec}^2 2\theta$

(C) Square root of the root of the equation $3x^{\log_3 4}+4^{\log_3 x}=64$ is equal to (R) 3

(D Suppose sin θ – cos θ = 1, then the value of $\sin^3\theta-\cos^3\theta$ is (Q ∈ R) (S) 4

(T) 5

4. **Column -I** **Column -II**

(A) In a triangle ABC, let **(P) 3**

$\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}=\sqrt{12\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$,

then the value of $\left(\frac{\sin^2 A}{\sin^2 B}+\frac{\sin^2 B}{\sin^2 C}+\frac{\sin^2 C}{\sin^2 A}\right)$ is

(B) Let A = $|(\log_2|\cos 24°) + (\log_2|\cos 48°|) + (\log_2|\cos 96°|) + (\log_2|\cos 192°|) + (\log_2|\cos 96°|) + (\log_2|\cos 192°|)|$ and **(Q) 5**

$$B=\frac{\log_{41}23\cdot\log_{21}56\cdot\log_{23}91\cdot\log_{56}41}{\log_{11}91\cdot\log_{21}11}$$

The value of (AB), is

(C) Let L denotes the value of $\cos^2(\alpha-\beta)$, if **(R) 6**

$\sin 2\alpha + \sin 2\beta = \frac{1}{2}$ and $\cos 2\alpha+\cos 2\beta = \frac{\sqrt{3}}{2}$.

If M denotes the value of expression

$\frac{1}{\log_{xy}xyz}+\frac{1}{\log_{yz}xyz}+\frac{1}{\log_{zx}xyz}$, **(S) 4**

5. Calculate without using trigonometric tables :

Column -I **Column -II**

(A) $\text{cosec}\,10° - \sqrt{3}\sec 10°$ **(P) 4**

(B) $4\cos 20° - \sqrt{3}\cot 10°$ **(Q) –1**

(C) $\frac{2\cos 40° - \cos 20°}{\sin 20°}$ **(R) $\sqrt{3}$**

(D) $2\sqrt{2}\sin 10°\left[\frac{\sec 5°}{2}+\frac{\cos 40°}{\sin 5°}-2\sin 35°\right]$

6. Let α, β be the solutions of the equation 3cos 2θ + 4 sin 2θ = 5 then match the following

Column – I **Column – II**

(A) tan α + tan β (P) 0

(B) tan (α + β) (Q) $\frac{4}{3}$

(C) tan (α – β) (R) $\frac{1}{4}$

(D) tan α · tan β) (S) 1

7. **Column – I**

(A) tan 20° · tan 40° · tan 60° · tan 80° =

(B) tan 9° – tan 27° – tan 63° + tan 81° =

(C) $4\left(\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4\frac{5\pi}{16}+\sin^4\frac{7\pi}{16}\right)=$

(D) $4\left(\cos^6\frac{\pi}{16}+\cos^6\frac{3\pi}{16}+\cos^6\frac{5\pi}{16}+\cos^6\frac{7\pi}{16}\right)=$

Column – II

(P) 3 (Q) 4 (R) 6 (S) 5

8. **Column – I** **Column – II**

(A) Minimum value of $5\sin^2\theta+4\cos^2\theta$ is (P) 2

(B) The maximum value of (Q) 4

$\cos^2\left(\frac{\pi}{3}-x\right)-\cos^2\left(\frac{\pi}{3}+x\right)$

(C) Minimum value of (R) $\frac{\sqrt{3}}{2}$

$\tan^2\theta + \cot^2\theta$ is

(D) Minimum value of $9\tan^2\theta + 4\cot^2\theta$ is — (S) 12

PART - IV INTEGER TYPE

1. If $X = \sin\left(\theta + \frac{7\pi}{12}\right) + \sin\left(\theta - \frac{\pi}{12}\right) + \sin\left(\theta + \frac{3\pi}{12}\right)$;

$Y = \cos\left(\theta + \frac{7\pi}{12}\right) + \cos\left(\theta - \frac{\pi}{12}\right) + \cos\left(\theta + \frac{3\pi}{12}\right)$

then $\frac{X}{Y} - \frac{Y}{X} = k\ \tan 2\theta$. where k equals.

2. $\sin^2 12° + \sin^2 21° + \sin^2 39° + \sin^2 48° - \sin^2 9° - \sin^2 18° =$

3. If $A + B + C = \pi$, then $\sum\left(\frac{\tan A}{\tan B \cdot \tan C}\right)$

$= \sum(\tan A) - k\sum(\cot A)$ where k equals.

4. If $\frac{\tan 3\theta}{\tan\theta} = 4$, then the value of $3\frac{\sin 3\theta}{\sin\theta}$ is.

5. If $\cos 5\theta = a\cos\theta + b\cos^3\theta + d$, then $|b + c| =$

6. The value of $\sum_{r=0}^{10}\cos^3\frac{r\pi}{3}$ is equal to $\frac{-a}{b}$ then the value of b is (where g.c.d of (a,b) is 1)

7. The value of cosec 10° cosec 50° – cosec 50° cos 70° – cosec 10° cosec 70° equal

8. The value of $\sqrt{\operatorname{cosec}^2 10^o + \operatorname{cosec}^2 50^o + \operatorname{cosec}^2 70^o}$ equal.

9. If $\frac{3 - \tan^2\frac{\pi}{7}}{1 - \tan^2\frac{\pi}{7}} = K\cos\frac{\pi}{7}$ then K equals.

10. If k is the smallest positive integer solution to $\tan 19x^o = \frac{\cos 96^o + \sin 96^o}{\cos 96^o - \sin 96^o}$ then middle digit of k is equal to

11. If $S = \sum_{n=1}^{89}\frac{1}{1 + (\tan n^o)^2}$ then $\frac{[S]}{11}$ equals. ([.] denotes G.I.F.)

12. Let $\tan\alpha \cdot \tan\beta = \frac{1}{\sqrt{2005}}$. If the value of $(1003 - 1002\cos 2\alpha)(1003 - 1002\cos 2\beta)$ is K then sum of digits of K is.

13. If K is the minimum value of the function,

$$f(x) = \frac{\sin x}{\sqrt{1-\cos^2 x}} + \frac{\cos x}{\sqrt{1-\sin^2 x}} + \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$$

as x varies over all number in the largest possible domain of f then | k | equals.

14. If $\frac{\sin\theta\sin 2\theta + \sin 3\theta\sin 6\theta + \sin 4\theta\sin 13\theta}{\sin\theta\cos 2\theta + \sin 3\theta\cos 6\theta + \sin 4\theta\cos 13\theta} = \tan(k\theta)$, then find k.

15. Find the exact value of the expression

$$\frac{\sin^2 34^o - \sin^2 11^o}{\sin 34^o\cos 34^o - \sin 11^o\cos 11^o}.$$

16. Two parallel chords of a circle of radius 4, which are on the same side of the centre, subtend angles of $\frac{2\pi}{5}$ and $\frac{4\pi}{5}$ respect at the centre. Find the perpendicular distance between the chords.

17. If $N = \prod_{k=1}^{36}\sin(10k)^o = \frac{\sin(36\times 5^o)}{\sin 5^o}\sin\left(\frac{10^o + 360^o}{2}\right) = 0$

$N = \prod_{r=1}^{7}\sec\left(\frac{r\pi}{15}\right)$ then find the value of $\log_2 N$.

18. Find the value of $\prod_{k=0}^{5}2\cos(2^k A)$, where $A = \frac{\pi}{65}$.

19. Let $p = \sum_{n=1}^{9}\sin^2\left(\frac{n\pi}{24}\right)$ and $q = \sum_{n=1}^{9}\cos^2\left(\frac{n\pi}{24}\right)$, then find the value of (p + q).

20. If $\sum_{r=1}^{88}\tan r^o\tan(r+1)^o = \cot^2 1^o - k$, where k is a prime number, then find the difference of the digits in k.

21. If $x^2 + y^2 = 4$ and m & M are the minimum and maximum value of expression $(1 - 2x^2)^2 + 4x^2y^2$, then find the value of $\left(\frac{M}{7} - 2m\right)$.

22. If $\sin\frac{2\pi}{15}+\sin\frac{4\pi}{15}+\sin\frac{8\pi}{15}+\sin\frac{16\pi}{15}$ has the value equal to $\sqrt{\frac{a}{b}}$ where a and b $\in$ N and are relatively prime, then find the value of (a + b).

23. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chord subtend at the centre, angles of $\frac{\pi}{K}$ and $\frac{2\pi}{K}$, where K > 0 then find the value of [K].[Notes : [K] denotes the largest integer less than or equal to K].

PART - V SUBJECTIVE TYPE

1. Prove that $\frac{\sin x-\cos x+1}{\sin x+\cos x-1}=\sec x+\tan x$.

2. If 2 cos x + sin x = 1, then find the value of 7 cos x + 6 sin x.

3. A parallelogram containing a 60º angle has perimeter p and its longer diagonal is of length d. Find its area.

4. If $u_n = \sin^n\theta + \cos^n\theta$, then prove that $\frac{u_5-u_7}{u_3-u_5}=\frac{u_3}{u_1}$.

5. Let A = sin x cos x. Then find the value of $\sin^4 x + \cos^4 x$ in terms of A.

6. If $\frac{\sec^4\theta}{a}+\frac{\tan^4\theta}{b}=\frac{1}{a+b}$, then prove that $|b| \le |a|$.

7. If ABC is a triangle and $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in H.P., then find the minimum value of cot B/2.

8. Find the sum of the series cosec θ + cosec 2θ + cosec 4θ +····· to n terms.

9. If tan 6q = p/q, find the value of $\frac{1}{2}$ (p cosec 2θ – q sec 2θ) in terms of p and q.

10. If 0 < α < π/2 and sin α + cos α + tan α + cot α + sec α + cosec α = 7, then prove that sin 2 α is a root of the equation $x^2-44x+36=0$.

11. Let A, B, C be three angles such that A = π/4 and tan B tan C = p. Find all possible values of p such that A, B, C are the angles of a triangle.

12. Show that

$$\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\cos 9x}{\cos 27x}=\frac{1}{2}[\tan 27x-\tan x]$$

13. Prove that $\frac{2\cos 2^n\theta+1}{2\cos\theta+1}$ = (2cos θ – 1) (2 cos 2θ – 1) × (cos 2^2 θ – 1) ..... (2 cos 2^{n-1} θ – 1).

14. Prove that $\frac{\tan 2^n\theta}{\tan\theta}=(1+\sec 2\theta)(1+\sec 2^2\theta)\times(1+\sec 2^3\theta)$ $\cdots(1+\sec 2^n\theta)$.

15. If $X=\sin\left(\theta+\frac{7\pi}{12}\right)+\sin\left(\theta-\frac{\pi}{12}\right)+\sin\left(\theta+\frac{3\pi}{12}\right)$,

$$Y=\cos\left(\theta+\frac{7\pi}{12}\right)+\cos\left(\theta-\frac{\pi}{12}\right)+\cos\left(\theta+\frac{3\pi}{12}\right)$$

then prove that $\frac{X}{Y}-\frac{Y}{X}=2\tan 2\theta$.

16. In triangle ABC, tan (A – B) + tan (B – C) + tan (C – A) = 0. Prove that the triangle isisosceles.

17. In a right angled triangle, acute angles A and B satisfy tan A + tan B + $\tan^2$ A + $\tan^2$ B $\tan^3$ B = 70. Find the angle A and B in radians.

EXERCISE - IV

1. If $\cos x + \cos y + \cos\alpha = 0$ and $\sin x + \sin y + \sin\alpha = 0$, then $\cot\left(\frac{x+y}{2}\right) =$ **[AIEEE : 2002]**

(A) $\sin\alpha$ (B) $\cos\alpha$ (C) $\cot\alpha$ (D) $2\sin\alpha$

2. $\cos 1^\circ \cos 2^\circ . \cos 3^\circ$ .... $\cos 179^\circ =$ **[AIEEE : 2002]**

(A) 0 (B) 1 (C) 2 (D) 3

3. Let α, β be such that $\pi < \alpha - \beta < 3\pi$. **[AIEEE : 2004]**

If $\sin\alpha + \sin\beta = -\frac{21}{65}$ & $\cos\alpha + \cos\beta = -\frac{27}{65}$, then the value of $\cos\frac{\alpha-\beta}{2}$ is :

(A) $-\frac{3}{\sqrt{310}}$ (B) $\frac{3}{\sqrt{130}}$

(C) $\frac{6}{65}$ (D) $\frac{-6}{65}$

4. Let $\cos(\alpha+\beta) = \frac{4}{5}$ and let $\sin(\alpha-\beta) = \frac{5}{13}$, where $0 \le \alpha, \beta \le \frac{\pi}{4}$. Then $\tan 2\alpha =$ **[AIEEE : 2010]**

(A) $\frac{25}{16}$ (B) $\frac{56}{33}$ (C) $\frac{19}{12}$ (D) $\frac{20}{7}$

5. If $A = \cos^2\theta + \sin^4\theta$, then for all value of θ

(A) $1 \le A \le 2$ (B) $\frac{3}{4} \le A \le 1$

(C) $\frac{13}{16} \le A \le 1$ (D) None of these

6. In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to **[AIEEE : 2012]**

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{3\pi}{4}$ (D) $\frac{5\pi}{6}$

7. For $0 < \phi < \pi/2$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\phi$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$, $z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$, then **[IIT : 1992]**

(A) $xyz = xz + y$ (B) $xyz = xy + z$

(C) $xyz = x + y + z$ (D) None of these

8. If $K = \sin(\pi/18)\sin(5\pi/18)\sin(7\pi/18)$, then the numerical value of K is : **[IIT : 1993]**

(A) 1/8 (B) 1/16 (C) 1/2 (D) None of these

9. If $A > 0, B > 0$ and $A + B = \pi/3$, then the maximum value of $\tan A \tan B$ is : **[IIT : 1993]**

(A) 1 (B) 1/3 (C) $\sqrt{3}$ (D) $1/\sqrt{3}$

10. The expression $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right) + \sin^4(3\pi-\alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right) + \sin^6(5\pi-\alpha)\right]$ is equal to

(A) 0 (B) 1

(C) 3 (D) $\sin 4\alpha + \cos 6\alpha$

[IIT : 1995]

11. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) =$

(A) 11 (B) 12 (C) 13 (D) 14

[IIT : 1995]

12. $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true, if and only if : **[IIT : 1996]**

(A) $x + y \ne 0$ (B) $x = y, x \ne 0$

(C) $x = y$ (D) $x \ne 0, y \ne 0$

13. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}\,x)$ attains its maximum is :

(A) 0 (B) 1 (C) 2 (D) Infinite

[IIT 1998]

14. Which of the following number(s) is rational :

(A) $\sin 15^\circ$ (B) $\cos 15^\circ$

(C) $\sin 15^\circ \cos 15^\circ$ (D) $\sin 15^\circ \cos 75^\circ$

[IIT 1998]

15. Let n be an odd integer.

If $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r\theta$, for every value of θ then :

(A) $b_0 = 1, b_1 = 3$ (B) $b_0 = 0, b_1 = n$

(C) $b_0 = -1, b_1 = n$ (D) $b_0 = 0, b_1 = n^2 + 3n + 3$

[IIT : 1998]

16. The function $f(x) = \sin^4 x + \cos^4 x$ increases if

(A) $0 < x < \frac{\pi}{8}$ (B) $\frac{\pi}{4} < x < \frac{3\pi}{8}$

(C) $\frac{3\pi}{4} < x < \frac{5\pi}{8}$ (D) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

[IIT 1999]

17. In a triangle PQR, $R = \frac{\pi}{2}$, If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of the equation $ax^2 + bx + c = 0$ $(a \ne 0)$, then : **[IIT 1999]**

(A) $a + b = c$ (B) $b + c = a$

(C) $a + c = b$ (D) $b = c$

18. For a positive integer n, let $f_n(\theta) = \tan\left(\frac{\theta}{2}\right)(1 + \sec\theta)(1 + \sec 2\theta)(1 + \sec 4\theta) ..... (1 + \sec 2^n\theta)$. **[IIT 1999]**
Then :

(A) $f_2\left(\frac{\pi}{16}\right) = 2$ (B) $f_3\left(\frac{\pi}{32}\right) = 1$

(C) $f_4\left(\frac{\pi}{64}\right) = 0$ (D) None of these

19. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then $f(\theta)$ **[IIT 2000]**
(A) ≥ 0 only when $\theta \geq 0$ (B) ≤ 0 for all real θ
(C) ≥ 0 for all real θ (D) ≤ 0 only when $\theta \leq 0$

20. If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then $\tan\alpha$ equals : **[IIT 2001]**
(A) $2(\tan\beta + \tan\gamma)$ (B) $\tan\beta + \tan\gamma$
(C) $\tan\beta + 2\tan\gamma$ (D) $2\tan\beta + \tan\gamma$

21. The maximum value of
$(\cos\alpha_1).(\cos\alpha_2)....(\cos\alpha_n)$, under the restrictions $0 \leq \alpha_1 . \alpha_2 ..... \alpha_n \leq \frac{\pi}{2}$ and $(\cot\alpha_1).(\cot\alpha_2).(\cot\alpha_3)......(\cot\alpha_n) = 1$ is

(A) $\frac{1}{2^{n/2}}$ (B) $\frac{1}{2^n}$ (C) $\frac{1}{2n}$ (D) 1

22. If θ & ϕ are acute angles such that $\sin\theta = \frac{1}{2}$ and $\cos\phi = \frac{1}{3}$ then $\theta + \phi$ lies in **[IIT 2004]**

(A) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right]$ (B) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

(C) $\left(\frac{2\pi}{3}, \frac{5\pi}{6}\right]$ (D) $\left(\frac{\pi}{6}, \pi\right]$

23. $\cos(\alpha + \beta) = \frac{1}{e}$, $\cos(\alpha - \beta) = 1$ find no. of ordered pair of $(\alpha, \beta) - \pi \leq \alpha, \beta \leq \pi$ **[IIT Scr 2005]**
(A) 0 (B) 1 (C) 2 (D) 4

24. If $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$, $t_4 = (\cot\theta)^{\cot\theta}$ and let $\theta \in \left(0, \frac{\pi}{4}\right)$, then **[IIT 2006]**

(A) $t_4 < t_2 < t_1 < t_3$ (B) $t_4 < t_1 < t_3 < t_2$
(C) $t_4 < t_3 < t_2 < t_1$ (D) $t_2 < t_1 < t_3 < t_4$

25. If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then **[IIT 2009]**

(A) $\tan^2 x = \frac{2}{3}$ (B) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(C) $\tan^2 x = \frac{1}{3}$ (D) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

26. The maximum value of the expression

$\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is **[IIT 2010]**

27. Let $\theta, \phi \in [0, 2\pi]$ be such that $2\cos\theta(1 - \sin\phi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\phi - 1$, $\tan(2\pi - \theta) > 0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$. Then ϕ cannot satisfy **[IIT 2012]**

(A) $0 < \phi < -\frac{\pi}{2}$ (B) $\frac{\pi}{2} < \phi < \frac{4\pi}{3}$

(C) $\frac{4\pi}{3} < \phi < \frac{3\pi}{3}$ (D) $\frac{3\pi}{2} < \phi < 2\pi$

28. Let $f : (-1, 1) \to IR$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ is(are)

(A) $1 - \sqrt{\frac{3}{2}}$ (B) $1 + \sqrt{\frac{3}{2}}$ **[IIT 2012]**

(C) $1 - \sqrt{\frac{2}{3}}$ (D) $1 + \sqrt{\frac{2}{3}}$

29. The expression $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$ can be written as
(A) secA cosecA + 1 (B) tanA + cotA
(C) secA + cosecA (D) sinA cosA + 1
[JEE Mains 2013]

30. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ where $x \in R$ and $k \geq 1$. Then $f_4(x) - f_6(x)$ equals **[JEE Mains 2014]**

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{12}$

31. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal to **[JEE Adv. 2016]**

(A) $3-\sqrt{3}$ (B) $2(3-\sqrt{3})$

(C) $2(\sqrt{3}-1)$ (D) $2(2+\sqrt{3})$

32. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is **[JEE Mains 2017]**

(A) $-\frac{3}{5}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $-\frac{7}{9}$

33. Let a vertical tower AB have its end A on the level ground. Let C be the mid - point of AB and P be a point on the ground such that AP = 2AB. If $\angle BPC = b$ then tan b is equal to : **[JEE Mains 2017]**

(A) $\frac{6}{7}$ (B) $\frac{1}{4}$ (C) $\frac{2}{9}$ (D) $\frac{4}{9}$

34. Let α and β be non zero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then which of the following is/are true ? **[JEE Adv. 2017]**

(A) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

(B) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(C) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

35. Let a, b, c be three non-zero real numbers such that the equation **[JEE Adv. 2018]**

$$\sqrt{3}\,a\cos x + 2b\sin x = c,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is.

MOCK TEST

SINGLE CORRECT CHOICE TYPE

1. If $k_1 = \tan 27\theta - \tan\theta$ and $k_2 = \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta}$, then

(A) $k_1 = 2k_2$ (B) $k_1 = k_2 + 4$
(C) $k_1 = k_2$ (D) None of these

2. If $\frac{\sec^8\theta}{a} + \frac{\tan^8\theta}{b} = \frac{1}{a+b}$, then for every real value of $\sin^2\theta$

(A) $ab \le 0$ (B) $ab \ge 0$
(C) $a + b = 0$ (D) None of these

3. If $0° < \theta < 180°$, then $\sqrt{2+\sqrt{2+\sqrt{2+....+\sqrt{2(1+\cos\theta)}}}}$ then being n number of 2's, is equal to

(A) $2\cos\left(\frac{\theta}{2^n}\right)$ (B) $2\cos\left(\frac{\theta}{2^{n-1}}\right)$

(C) $2\cos\left(\frac{\theta}{2^{n+1}}\right)$ (D) None of these

4. The value of the expression

$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{10\pi}{7} - \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is equal to

(A) 0 (B) $-\frac{1}{4}$ (C) $\frac{1}{4}$ (D) $-\frac{1}{8}$

5. The value of $e^{\log_{10}\tan 1° + \log_{10}\tan 2° + \log_{10}\tan 3° + ... + \log_{10}\tan 89°}$ is equal to

(A) 0 (B) e (C) 1/e (D) None of these

MULTIPLE CORRECT CHOICE TYPE

6. Let $P_n(u)$ be a polynomial in u of degree n. Then, for every positive integer n, sin 2nx is expressible is

(A) $P_{2n}(\sin x)$ (B) $P_{2n}(\cos x)$
(C) $\cos x\, P_{2n-1}(\sin x)$ (D) $\sin x\, P_{2n-1}(\cos x)$

7. $\frac{3 + \cos 76° \cot 16°}{\cot 76° + \cot 16°}$ is equal to

(A) tan 16° (B) cot 76° (C) tan 46° (D) cot 44°

8. If $x = \sec\phi - \tan\phi$ and $y = \text{cosec}\,\phi + \cot\phi$, then

(A) $x = \frac{y+1}{y-1}$ (B) $x = \frac{y-1}{y+1}$

(C) $y = \frac{1+x}{1-x}$ (D) $xy + x - y + 1 = 0$

9. In a ΔABC

(A) $\sin A \sin B \sin C \le \frac{3\sqrt{3}}{8}$

(B) $\sin^2 A + \sin^2 B + \sin^2 C \le \frac{9}{4}$

(C) sin A sin B sin C is always positive

(D) $\sin^2 A + \sin^2 B \le 1 + \cos C$

10. If $\frac{x}{y} = \frac{\cos A}{\cos B}$, where $A \ne B$, then

(A) $\tan\left(\frac{A+B}{2}\right) = \frac{x\tan A + y\tan B}{x+y}$

(B) $\tan\left(\frac{A-B}{2}\right) = \frac{x\tan A - y\tan B}{x+y}$

(C) $\frac{\sin(A+B)}{\sin(A-B)} = \frac{y\sin A + x\sin B}{y\sin A - x\sin B}$

(D) $x\cos A + y\cos B = 0$

SUBJECTIVE

11. Let the exact value of expression

$E = \cos\frac{\pi}{7} + \cos^2\frac{\pi}{7} - 2\cos^3\frac{\pi}{7}$ is a rational number in the form p/q, where p and q are integers find (p + q).

12. For $A \in \left(0, \frac{\pi}{4}\right)$ if $\left(1 + \frac{\cos 3A}{\cos A}\right) + \left(1 + \frac{\cos 6A}{\cos 2A}\right)$

$+\left(1 + \frac{\cos 9A}{\cos 3A}\right) + \left(1 + \frac{\cos 12A}{\cos 4A}\right) = 0$

then find the value of (cosec A – sec 2A)

13. If $(4\cos^2 40° - 3)(3 - 4\sin^2 40°) = a + b\cos 20°$, find (a – b)

14. Let $\alpha = 4\sin^2 10° + 4\sin^2 50° . \cos 20° + \cos 80°$ and

$\beta = \cos^2\frac{\pi}{5} + \cos^2\frac{2\pi}{15} + \cos^2\frac{8\pi}{15}$, find $(\alpha + \beta)$

15. If $\sin 25° \cdot \sin 35° \cdot \sin 85° = \frac{\sqrt{a} + \sqrt{b}}{c}$, where $a, b, c \in N$.

Find (a + b + c).

INTEGER

16. Let $f = \sec 0 + \sec\frac{\pi}{7} + \sec\frac{2\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{4\pi}{7} + \sec\frac{5\pi}{7} +$ $\sec\frac{6\pi}{7}$. and $c = \cot 1° + \cot 2° + \cot 3° + ... + \cot 177° + \cot 178° + \cot 179°$. Find $(f^2 + c^2)$

17. Find the exact value of the expression

$$\frac{\sin^2 34° - \sin^2 11°}{\sin 34° \cos 34° - \sin 11° \cos 11°}$$

18. Find the value of $\prod_{k=0}^{5} 2\cos(2^k A)$, when $A = \frac{\pi}{65}$.

19. If $\sum_{r=1}^{88} \tan r° \tan (r+1)° = \cot^2 1° - k$, where k is prime number, then the difference of the digits in k.

20. If the maximum and minimum value of $(\sin x - \cos x - 1)(\sin x + \cos x - 1)\ \forall\ x \in R$ is M and m then find value of $(M - 4m)$

DPP 1

2. $\tan\theta = \dfrac{p^2-1}{2p}$ **3.** $\pm \cos A \cos B \cos C$

4. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$ **5.** 10

6. 0.79 rad **7.** 68°43'37.8"

8. $s = \dfrac{5\pi}{12}$ cm **9.** 25°12'

10. 5 : 4

DPP 2

2. $\sin\theta = \dfrac{-12}{13}$, the angle 'θ' lies in the 4^{th} quadrant.

3. **(A)** **4.** **(A)**

5. **(B)** **7.** $\dfrac{\sqrt{7}}{4}$

9. **(C)** **10.** $-n$

11. $x \in [1, 2]$ **16.** 1

DPP 3

4. 1 **5.** $\sin 3A = -59/72$.

6. $\cos\theta = \left(\sum \cos x\right) + \sin\theta\left(\sum \sin x\right) = 0$

7. $\tan\alpha = -1$

DPP 4

6. 11

DPP 6

3. cos7° **4.** $\dfrac{1}{2}$

5. $\dfrac{\pi}{2}$

DPP 7

4. $y \in [0, \sqrt{3}]$

5. $y \in [-1, -1/3]$

6. $-1/4 \le (\cos^2 x - 1/2)^2 - 1/4 \le 0$

8. Hence, the maximum value is $1 + \sqrt{5}$

9. $y \in [-5, 5]$

10. $1 \le p(x) \le 11$

EXERCISE - 1

1. (C)	**2.** (A)	**3.** (A)	**4.** (B)	**5.** (A)	**6.** (B)	**7.** (C)	**8.** (C)	**9.** (B)	**10.** (B)
11. (A)	**12.** (D)	**13.** (A)	**14.** (B)	**15.** (A)	**16.** (C)	**17.** (B)	**18.** (B)	**19.** (D)	**20.** (B)
21. (D)	**22.** (A)	**23.** (B)	**24.** (A)	**25.** (D)	**26.** (B)	**27.** (C)	**28.** (B)	**29.** (B)	**30.** (C)
31. (A)	**32.** (A)	**33.** (C)	**34.** (A)	**35.** (B)	**36.** (B)	**37.** (B)	**38.** (D)	**39.** (B)	**40.** (B)
41. (A)	**42.** (B)	**43.** (B)	**44.** (D)	**45.** (D)	**46.** (A)	**47.** (A)	**48.** (C)	**49.** (A)	**50.** (A)
51. (B)	**52.** (B)	**53.** (B)	**54.** (C)	**55.** (A)	**56.** (B)	**57.** (D)	**58.** (C)	**59.** (C)	**60.** (D)

EXERCISE - 2

1. (A, C)	**2.** (A, B, C)	**3.** (A, D)	**4.** (A, B, C, D)	**5.** (B, C)	**6.** (C, D)	**7.** (A, B, C)
8. (A, C, D)	**9.** (A)	**10.** (B, C)	**11.** (A, B, D)	**12.** (B, C, D)	**13.** (B, C)	**14.** (A, B, C, D)
15. (A, B, C, D)	**16.** (B, C)	**17.** (B, C, D)	**18.** (C, D)	**19.** (A, D)	**20.** (B, D)	**21.** (B, D)
22. (A, C, D)	**23.** (A, B, C, D)	**24.** (A, B, D)	**25.** (A, B, C)	**26.** (A, C)	**27.** (A, D)	**28.** (B, D)
29. (B, D)						

EXERCISE - 3

PART - I

Assertion Reason Type

1. (B)	**2.** (C)	**3.** (D)	**4.** (A)	**5.** (B)	**6.** (A)	**7.** (A)	**8.** (B)	**9.** (B)	**10.** (A)
11. (B)	**12.** (A)	**13.** (C)	**14.** (D)	**15.** (D)	**16.** (A)	**17.** (D)	**18.** (A)		

PART - II

Comprehension Type

Comprehension 1

1. (D) **2.** (A) **3.** (C)

Comprehension 2

1. (B) **2.** (A) **3.** (D)

Comprehension 3

1. (A) **2.** (D)

Comprehension 4

1. (A) **2.** (B)

Comprehension 5

1. (A) **2.** (A)

Comprehension 6

1. (A) **2.** (B)

Comprehension 7

1. (A) **2.** (D)

Comprehension 8

1. (B) **2.** (B) **3.** (A)

Comprehension 9

1. (A) **2.** (C) **3.** (B)

Comprehension 10

PART - III

Match the Column Type

1. (A) $\rightarrow$ P, Q ; (B) $\rightarrow$ P, R, T; (C) $\rightarrow$ P, Q, R, S, T ; (D) $\rightarrow$ T

2. **(A) $\rightarrow$ P, Q; (B) $\rightarrow$ P, Q, R, S ; (C) $\rightarrow$ Q**

3. (A) $\rightarrow$ Q; (B) $\rightarrow$ T; (C) $\rightarrow$ R; (D) $\rightarrow$ P

4.

5. (A) $\rightarrow$ P ; (B) $\rightarrow$ Q, (C) $\rightarrow$ R ; (D) $\rightarrow$ S

6. (A) $\rightarrow$ S ; (B) $\rightarrow$ Q, (C) $\rightarrow$ P ; (D) $\rightarrow$ R

7. (A) $\rightarrow$ P ; (B) $\rightarrow$ Q, (C) $\rightarrow$ R ; (D) $\rightarrow$ P

8. (A) $\rightarrow$ Q ; (B) $\rightarrow$ R, (C) $\rightarrow$ P ; (D) $\rightarrow$ S

PART - IV

Integer Type

1. (2)	**2.** (1)	**3.** (2)	**4.** (8)	**5.** (4)	**6.** (8)	**7.** (0)	**8.** (6)	**9.** (4)	**10.** (5)
11. (4)	**12.** (2005)	**13.** (–2)	**14.** (0009)	**15.** (1)	**16.** (2)	**17.** (0007)	**18.** (1)	**19.** (9)	**20.** (1)
21. (5)	**22.** (0019)	**23.** (3)							

PART - V

Subjective Type

EXERCISE - 4

1. (C)	**2.** (0)	**3.** (A)	**4.** (B)	**5.** (B)	**6.** (A)	**7.** (B, C)	**8.** (A)	**9.** (B)	**10.** (B)
11. (C)	**12.** (B)	**13.** (A)	**14.** (C)	**15.** (B)	**16.** ( )	**17.** ( )	**18.** (B)	**19.** (C)	**20.** (B)
21. (A)	**22.** (B)	**23.** (D)	**24.** (D)	**25.** (A, B)	**26.** (2)	**27.** (A, C, D)	**28.** (A, B)	**29.** (A)	**30.** (D)
31. (C)	**32.** (D)	**33.** (C)	**34.** (B, C)	**35.** (0.5)					

EXERCISE –III

PART - I ASSERTION REASON

1. **(B)**

2. **(C)**

3. **(D)**

x sin θ. $\sin^2 \theta$ + y cos ⇒. $\cos^2$⇒ = sin θ cos θ

⇒ x sin θ = sin θ cos θ ⇒ x = cos θ ⇒ y = sin θ

$x^2 + y^2 = 1$. so statement 1 is true. Also statement 2 is true but does not imply statement 1.

4. **(A)**

5. **(B)**

6. **(A)**

7. **(A)**

$\because$ AM ≥ GM $\because \dfrac{x+\frac{1}{x}}{2} \ge 1$

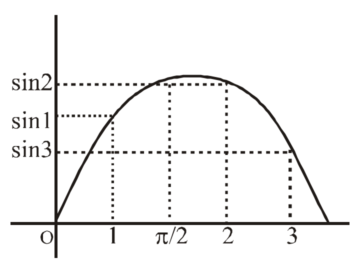
⇒ sin θ ≥ 2 which is impossible. ($\because$ –1 ≤ sin θ ≤ 1)

8. **(B)**

9. **(B)**

From figure, clearly

sin3 < sin1 < sin2



10. **(A)**

Let A = $\begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix} = (3\cos\theta - \sin\theta)^2$

$\therefore \quad \Delta_{max} = 10$

11. **(B)**

cos 7 = cos (2π + 7 – 2 π) = cos (7 – 2π) = cos (0. 72)

Now 1 rad and 0.72 rad angles are for the first quadrant where cos x is decreasing; hence,

cos1 < cos 0. 72 or cos 1 < cos 7.

But statement 2 is not the correct explanation for cos 1 < cos 7. note that cos 0.5 > cos 7.

12. **(A)**

$f(\theta) = (\sin\theta + \text{cosec}\,\theta)^2 + (\cos\theta + \sec\theta)^2$

$= \sin^2\theta + \cos^2\theta + \text{cosec}^2\theta + \sec^{22} + 4$

$= 5 + \dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta} = 5 + \dfrac{1}{\sin\theta\cdot\cos^2\theta} \ge 9$

Now $(\sin\theta - \cos\theta)^2 \ge 0$

$\therefore \quad \sin^2\theta + \cos^2\theta \ge 2\sin\theta\cos\theta$

$\therefore \quad \dfrac{1}{4} \ge \sin^2\theta\cos^2\theta$

$\therefore \quad \dfrac{1}{\sin^2\theta\cos^2\theta} \ge 4$

Hence, the correct answer is (a).

13. **(C)**

Clearly, statement 2 is false.

Statement 1 is true as select α = 2π, β = –π/2, γ = – π/2

Then sin α + sin β + sin γ = 0 – 1 – 1 = –2, which shows that the minimum value will be negative.

14. **(D)**

Statement 1 is wrong as z can be written as $\dfrac{-(x+y)}{1-xy}$.

It implies that for any values of xy (xy ≠ 1), we get a value of z and statement is correct.

15. **(D)**

The value of $\sin\sqrt{2}x + \sin ax$ can be equal to 2, if $\sin\sqrt{2}x$ and sin ax both are equal to 1 but they are not equal to 1 for any common value of x.

16. **(A)**

Statement 2 is true as it is one of the standard results of multiple angles.

Putting A = π/18 in the formula $\sin 3A = 3\sin A - 4\sin^3 A$, we get $8x^3 - 6x + 1 = 0$, where x = sin π/18. Hence, statement 1 is also true because of statement 2.

17. **(D)**

Statement is true as it is one of the conditional identies in the triangle. Since R.H.S. > 1 in statement 2, statement 1 is false.

18. **(A)**

$\cos^2 A\cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$

$\therefore \quad \sum \cos^2 A\,|_{min} = 1 - 2\times\dfrac{1}{8}| = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

PART - II COMPREHENSION TYPE

Comprehension # 1

1. **(D)** **2.** **(A)** **3.** **(C)**

$\cos^2 63° + \cos^2 57° + \cos 63° \cdot \cos 57°$

$= 1 - \sin^2 63° + \cos^2 57° + \cos 63° \cdot \cos 57°$

$= 1+\cos 120^o \cdot \cos 6^o + \frac{1}{2}\left(\cos 120^o + \cos 60^o\right)$

$= 1-\frac{1}{2}\cos 6^o + \frac{1}{2}\left(-\frac{1}{2}\right)+\frac{1}{2}\cos 6^o$

$= 1-\frac{1}{4} = \frac{3}{4} \Rightarrow \log_{1.\bar{3}}\left(\frac{3}{4}\right) = \log_{4/3}\left(\frac{3}{4}\right) = -1$

Now $\frac{8}{9}P_2(x) = \frac{8}{9}\cdot\frac{1}{2} = \frac{4}{9}$

Hence $\left(\frac{8}{9}P_2(x)\right)^{100} = \left(\frac{4}{9}\right)^{100} = y(\text{say})$

$\log y = 200[\log 2 - \log 3] = -35.62 = \overline{36}.38 \Rightarrow 35$

zero's ans (iii)

Now $12(P_4(x) - P_6(x))$

$= 3(\sin^4 x + \cos^4 x) - 2(\sin^6 x + \cos^6 x)$

$= 3[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cdot \cos^2 x] - 2[\sin^2 x + \cos^2 x)$
$(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)]$

$= 3[1 - 2\sin^2 x \cdot \cos^2 x] - 2[1 - 3\sin^2 x \cos^2 x]$

$= 3 - 6\sin^2 x \cdot \cos^2 x - 2 + 6\sin^2 x \cos^2 x = 1$

Comprehension # 2

1. (B) 2. (A) 3. (D)

(i) $\sum a^2 = \frac{1}{2}[3+\cos 20^o + \cos 100^o + \cos 140^o]$

$= \frac{1}{2}[3+\cos 20^o - \left(\cos 80^o + \cos 40^o\right)$

$= \frac{1}{2}[3+\cos 20^o - 2\cos 60^o \cos 20^o] = \frac{3}{2}$

(ii) $b + c = \cos 50° + \cos 70° = 2\cos 60° \cos 10°$

$= \cos 10° = a$

(iii) $(b + c)^2 = a^2$ .....(i)

$b - c = \cos 50° - \cos 70° = 2\sin 60° \sin 10°$

$= \sqrt{3}\sin 10^o$

$(b - c)^2 = 3\sin^2 10° = 3(1 - a^2)$ .....(ii)

(i) & (ii)

$\Rightarrow 4bc = 4a^2 - 3 \Rightarrow 4(a^2 - bc) = 3.$

Comprehension # 3

1. (A)

2. (D)

$\cos(2\theta) = \sqrt{\frac{2\left(5+\sqrt{5}\right)}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

$= \cos 18°$ or $\cos(-18°)$

$\Rightarrow \theta = 9° \in (-45°, 45°)$

Comprehension # 4

1. (A) 2. (B)

$\sin\alpha + \sin\beta = \frac{\sqrt{6}}{3}$ .....(i)

$\cos\alpha + \cos\beta = \frac{\sqrt{3}}{3}$ .....(ii)

$\therefore$ Divide (i) by (ii), $\tan\left(\frac{\alpha+\beta}{2}\right) = \sqrt{2}$

Squaring and adding equation (i) and (ii), we get

$2 + 2\cos(\alpha - \beta) = 1 \Rightarrow 1 + \cos(\alpha - \beta) = \frac{1}{2}$

$\therefore 100\cos^2\left(\frac{a-b}{2}\right) = 50\left(1+\cos\left(\alpha-\beta\right)\right) = 25$

Comprehension # 5

1. (A) 2. (A)

(i) $P(x) = (x - \tan 10°)(x - \tan 50°)(x - \tan 70°)$

$\therefore$ Constant term $= -\tan 10° \tan 50° \tan 70°$ (put $x = 0$)

$= -\tan 30° = \frac{-1}{\sqrt{3}}$

(ii) $a = -\tan(10° + \tan 50° + \tan 70°)$

$b = (\tan 10° + \tan 50° + \tan 70°)$

$c = -\tan 10° \tan 50° \tan 70°$

$\therefore \frac{-a+c}{1-b}$

$= \frac{\left(\tan 10^o + \tan 50^o + \tan 70^o\right) - \tan 10^o \tan 50^o \tan 70^o}{1-\left(\tan 10^o \tan 50^o + \tan 50^o \tan 70^o + \tan 70^o \tan 10^o\right)} =$

$\tan(10° + 50° + 70°) = \tan 130°$

Comprehension # 6

For $n \in N$ let $f(n) = 8(\cos^n\theta - \sin^n\theta)$.

$g(n) = 8(\cos^n\theta + \sin^n\theta)$

1. (A)

2. (B) (i)

$f(6) = 8(\cos^6\theta - \sin^6\theta)$

$= 8(\cos^2\theta - \cos^2\theta)(\cos^4\theta + \sin^4\theta\cos^2\theta)$

$= 8\cos 2\theta\,(1 - \sin^2\theta\cos^2\theta)$

$$= 8\cos 2\theta\left(1 - \frac{\sin^2 2\theta}{4}\right)$$

$$= 2\cos 2\theta\left(4 - \frac{1-\cos 4\theta}{2}\right)$$

$= \cos 2\theta(8 - 1\cos 4\theta)$

$= 7\cos 2\theta + \cos 4\theta\cos 2\theta$

$$= 7\cos 2\theta + \frac{\cos 6\theta + \cos 2\theta}{2}$$

$$= \frac{15}{2}\cos 2\theta + \frac{1}{2}\cos 2\theta$$

$\therefore\quad a + b = 8$

(ii) $\dfrac{g(6)}{32} + \dfrac{3}{16}\sin^2 2\theta$

$$= \frac{8(\cos^6\theta + \sin^6\theta)}{32} + \frac{3}{16}\sin^2 2\theta$$

$$= \frac{1}{4}\left(1 - \frac{3}{4}\sin^2 2\theta\right) + \frac{3}{16}\sin^2 2\theta = \frac{1}{4}$$

Comprehension # 7

1. (A)

2. (D)

$(\log_{\tan x}\sin x)(\log_{\cos x}\tan x) = 1$

$\Rightarrow\ \log_{\cos x}\sin x = 1$

$\Rightarrow\ \sin x = \cos x$

$\therefore\ \tan x = 1$ (rejected)

$\Rightarrow\ N = 0$

$2\log_8(2x) + \log_8(x-1) = \dfrac{4}{3}$

$\Rightarrow\ (2x(x-1))^2 = 16$

$\Rightarrow\ x(x-1) = 2, -2$

$\Rightarrow\ x^2 - x - 2 = 0,\ x^2 - x + 2 = 0$

$\Rightarrow\ x = 2$

$\Rightarrow\ M = 2$

$\Rightarrow\ (8\cos 2\,9° - 6)(12\cos^2 27° - 9) = 6\tan y°$

$$= \frac{\cos 9^o\left(8\cos^2 9^o - 6\right)\left(12\cos^2 27^o - 9\right)}{\cos 9^o}$$

$$= 6\frac{\cos 81^o}{\cos 9^o} = 6\tan 9^o$$

$\therefore\quad P = 9$

(i) $M^2 + N^2 = 4$

(ii) $P + M = 11$

Comprehension # 8

1. (B) 2. (B) 3. (A)

Given equation is $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$

$$\Rightarrow \frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = \sin^2\alpha + \cos^2\alpha$$

$$\Rightarrow \frac{\sin^4\alpha}{\sin^2\beta} + \sin^2\alpha = \cos^2\alpha - \frac{\cos^4\alpha}{\cos^2\beta}$$

$$\Rightarrow \sin^2\alpha\left[\frac{\sin^2\alpha - \sin^2\beta}{\sin^2\beta}\right] = \cos^2\alpha\left[\frac{\cos^2\beta - \cos^2\alpha}{\cos^2\beta}\right]$$

$$\frac{\sin^2\alpha\left(\sin^2\alpha - \sin^2\beta\right)}{\sin^2\beta} - \frac{\cos^2\alpha\left(\sin^2\alpha - \sin^2\beta\right)}{\cos^2\beta} = 0$$

$\Rightarrow\ (\sin^2\alpha - \sin^2\beta)\,[\sin^2\alpha\cos^1\beta - \cos^2\alpha\sin^2\beta] = 0$

$\Rightarrow\ (\sin^2\alpha - \sin^2\beta)$

$[\sin^2\alpha\,(1 - \sin^2\beta] - (1 - \sin^2\alpha)\sin^2\beta] = 0$

$\Rightarrow\ (\sin^2\alpha - \sin^2\beta)^2 = 0 \Rightarrow\ \sin^2\alpha = \sin^2\alpha$

$\Rightarrow\ \cos^2\alpha = \cos^2\beta$

$$\frac{\sin^2\alpha}{\sin^2\beta} + \frac{\sin^2\beta}{\sin^2\alpha} = 2$$

$$\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\beta}{\cos^2\alpha} = 2$$

$$\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = \cos^2\beta\cdot\frac{\cos^2\beta}{\cos^2\alpha} + \sin^2\beta\frac{\sin^2\beta}{\sin^2\alpha}$$

$= \cos^2\beta + \sin^2\beta = 1$

Comprehension # 9

1. (A) 2. (C) 3. (B)

1. Coefficients of $x^2 = -(\cos 36° + \cos 84° + \cos 156°)$

$\cos(60° - 24°) + \cos(60° + 24°) - \cos 24° = 2\cos 60°\cos 24°$
$- \cos 24° = \cos 24° - \cos 24° = 0$

2. The coefficient of

$x = \cos 36° \cos 84° + \cos 84° \cos 156° + \cos 36° \cos 156°$

$= \cos(60° - \theta)\cos(60° + \theta) - \cos(60° + \theta)\cos\theta - \cos(60° - \theta)$

(Let $\theta = 24°$)

$= \cos^2 60° - \sin^2\theta - \cos\theta(\cos(60° + \theta) + \cos(60° - \theta))$

$= \cos^2 60° - \sin^2\theta - 2\cos\theta\cos 60°\cos\theta$

$= \cos^2 60 - 1 = \frac{1}{4} - 1 = -\frac{3}{4}$

3. Absolute term in $p(x) = \cos(60° - \theta)\cos(60° + \theta)\cos\theta$

$= \frac{\cos(3\cdot 24)}{4} = \frac{\cos 72^o}{4} = \frac{\sqrt{5}-1}{4\cdot 4} = \frac{\sqrt{5}-1}{16}$

as $\cos 72° = \frac{\sqrt{5}-1}{4}$

PART - III MATCH THE COLUMN

1.. (A) → P, Q ; (B) → P, R, T; (C) → P, Q, R, S, T ; (D) → T

(A) Given $(\text{cosec}\,\theta + \sin\theta) = 3$ .....(i)

$\Rightarrow$ $(\text{cosec}^2\theta + \sin\theta) = 3^2 - 2 = 7$ .....(ii)

$\therefore$ $(\text{cosec}^2\theta + \sin\theta)(\text{cosec}\,\theta\ \ \sin\theta) = 3 \times 7$

$\Rightarrow$ $(\text{cosec}^2\theta + \sin^3\theta) + (\text{cosec}\,\theta + \sin\theta) = 21$

$\Rightarrow$ $(\text{cosec}^3\theta + \sin^3\theta)\ 18$ .....(iii)

Now, $(\text{cosec}^3\theta + \sin^3\theta) \times (\text{cosec}^2\theta + \sin^2\theta) = 18 \times 7$

$\Rightarrow$ $(\text{cosec}^5\theta + \sin^5\theta) + (\text{cosec}\,\theta + \sin\theta) = 18 \times 7$

$\Rightarrow$ $(\text{cosec}^5\theta + \sin^5\theta) = 126 - 3 = 123.$

(B) Let expression,

$$= \cos^2\frac{\pi}{18} + \cos^2\frac{\pi}{9} + \cos^2\frac{7\pi}{18} + \cos^2\frac{4\pi}{9}$$

$$= \left(\cos^2\frac{\pi}{18} + \cos^2\frac{4\pi}{9}\right) + \left(\cos^2\frac{\pi}{9} + \cos^2\frac{7\pi}{18}\right)$$

$$= \left(\cos^2\frac{\pi}{18} + \sin^2\frac{\pi}{18}\right) + \left(\cos^2\frac{\pi}{9} + \sin^2\frac{\pi}{9}\right) = 1 + 1 = 2$$

(C) $5^{(\log_5 7)^{2x}} = 7^{(\log_7 5)^x}$

$\Rightarrow$ $5^{\log_5 7(\log_5 7)^{2x-1}} = 7^{(\log_7 5)^x}$

$\Rightarrow$ $7^{(\log_5 7)^{2x-1}} = 7^{(\log_7 5)^x}$

$\Rightarrow$ $(\log_5 7)^{2x-1} = (\log_7 5)^x$

$\Rightarrow$ $(\log_5 7)^{2x-1} = \frac{1}{(\log_5 7)^x}$

$\therefore$ $(\log_5 7)^{3x-1} = 1$

$\therefore$ $3x - 1 = 0 \Rightarrow x = \frac{1}{3}$

Aliter : Take log on both the sides.

(D) $\log(xy^3) = 1\ \log(x^2y) = 1$

$xy^3 = 10$ .....(i)

$x^2 y = 10$ .....(ii)

(i) ÷ (ii)

$y^2 = x$

$y^5 = 10$ $\Rightarrow$ $y = 10^{1/5}$

$x = 10^{2/5}$

$\log(xy) = \log 10^{3/5} = \frac{3}{5}$

2. (A) → P, Q; (B) → P, Q, R, S ; (C) → Q

(A) $\cos^2 x + 5\cos x = 2 - 2\cos^2 x$

$3\cos^2 x + 5\cos x - 2 = 0$

$3\cos^2 x + 6\cos x - \cos x - 2 = 0$

$3\cos x(\cos x + 2) - (\cos x + 2) = 0$

$\cos x = -2$ or $\cos x = \frac{1}{3}$

$\therefore$ $\sec x = 3$

(B) $(0.2)\left(1 + 9^{\log_3 8}\right)^{\log_{65} 5}$

$= (0.2)\left(1 + 3^{\log_3 64}\right)^{\log_{65} 5} = (0.2)(1 + 64)^{\log_{65} 5}$

$= (0.2)(65)^{\log_{65} 5} = (0.2)(5) = 1$

(C) We have $(18\cos x + 9\sin x) + \sin x + a\sin x + b\cos x$

$\therefore$ $a = 9, b = 18$

$\therefore$ $b/a = 2$

Aliter : put $x = 0$ and $x = \frac{\pi}{2}$ to get a and b directly.

3. (A) → Q; (B) → T; (C) → R; (D) → P

(A) $\frac{\sin 50^o}{\cos 50^o} - \frac{\sin 40^o}{\cos 40^o} = k\frac{\sin 10^o}{\cos 10^o}$

$\Rightarrow$ $\frac{\sin 10^o \cos 10^o}{\sin 10^o[\cos 50^o \cos 40^o]} = k$

$\therefore$ $k = \frac{2\cos 10^o}{\cos 90^o + \cos 10^o} = 2$

(B) (i) $\sec^2\theta + \text{cosec}^2\theta = \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}$

$= \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta} = \dfrac{1}{\cos^2\theta} \times \dfrac{1}{\sin^2\theta}$

$= \sec^2\theta \cdot \text{cosec}^2\theta$

(ii) $(\tan\theta + \cot\theta)^3$

$= \left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^2 = \left(\dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)^2$

$= (\sec\theta \text{ cosec }\theta)^2 = \sec^2\theta \text{ cosec}^2\theta$

(iii) $\dfrac{\tan^2\theta + 1}{1 - \cos^2\theta} = \dfrac{\sec^2\theta}{\sin^2\theta} = \sec^2\theta \times \dfrac{1}{\sin^2\theta}$

$= \sec^2\theta \text{ cosec}^3\theta$

(iv) $\dfrac{1}{\sin^2 2\theta} = \dfrac{1}{4\sin^2\theta\cos^2\theta} = \dfrac{\sec^2\theta\,\text{cosec}^2\theta}{4}$

Hence number of expression = 3

(C) We have $3x^{\log_3 4} + 4^{\log_3} = 64$

$\Rightarrow\ 3 \cdot 4^{\log_3 x} + 4^{\log_3 x} = 64 \ \Rightarrow\ 4.4^{\log_3 x} = 64$

$\therefore\ 4^{\log_3 x} = 16 \qquad \Rightarrow \log_3 x = 2$

$\therefore\ x = 9$

$\Rightarrow$ Square root of the root of the given equation = 3.

(D) $\sin\theta - \cos\theta = 1 \Rightarrow 1 - 2\sin\theta \quad \cos q = 1$

$\therefore\ \sin\theta\cos\theta = 1$

Now, $\sin^3\theta - \cos^3\theta = (\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta) = (1)(1 + 0) = 1.$

4. (A) In ΔABC, let

$\left(\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C}\right) = \sqrt{12 \cdot \cos\dfrac{A}{2} \cdot \cos\dfrac{B}{2} \cdot \cos\dfrac{C}{2}}$

$\Rightarrow \left(\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C}\right) = \sqrt{3(\sin A + \sin B + \sin C)}$

Now, squaring on b.t.s. we get

$\sin A + \sin B + \sin C + 2\sqrt{\sin A \cdot \sin B}$

$+ 2\sqrt{\sin B \cdot \sin C} + 2\sqrt{\sin C \cdot \sin A}$

$= 3\sin A + 3\sin B + 3\sin C$

Which is possible when $\sin A = \sin B = \sin C$

$\Rightarrow\ \Delta ABC$ is equilibrium

$\therefore\ A = B = C = 60°$ (Each)

Hence the expression

$E = \dfrac{\sin^2 A}{\sin^2 B} = \dfrac{\sin^2 B}{\sin^2 C} + \dfrac{\sin^2 C}{\sin^2 A} = 1 + 1 + 1 = 3$

$= \tan 6\theta = \tan\left(6 \cdot \dfrac{\pi}{18}\right)$

$\Rightarrow\ \tan\dfrac{\pi}{3} = \sqrt{3} = E$

$\Rightarrow\ E^2 = 3$

5. (A) → P ; (B) → Q, (C) → R ; (D) → S

(B) $4\cos 20° - \sqrt{3}\cot 20°$

$= \dfrac{4\cos 20°\sin 20° - \sqrt{3}\cos 20°}{\sin 20°}$

$= \dfrac{2\sin 40° - \sqrt{3}\cos 20°}{\sin 20°}$

$= \dfrac{2\sin(60° - 20°) - \sqrt{3}\cos 20°}{\sin 20°}$

$= \dfrac{\sqrt{3}\cos 20° - \sin 20° - \sqrt{3}\cos 20°}{\sin 20°} = -1$

(C) $\dfrac{2\cos 40° - \cos 20°}{\sin 20°} = \dfrac{2\cos(60° - 20°) - \cos 20°}{\sin 20°}$

$= \dfrac{\cos 20° = \sqrt{3}\sin 20° - \cos 20°}{\sin 20°} = \sqrt{3}$

(D) $2\sqrt{2}\sin 10°\left[\dfrac{\sec 5°}{2} + \dfrac{\cos 40°}{\sin 5°} - 2\sin 35°\right]$

$= 2\sqrt{2}\left[\sin 5° + 2\cos 5°\cos 40° - 2\sin 10°\sin 35°\right]$

$= 2\sqrt{2}\left[\sin 5° + \cos 45°\cos 35° - (\cos 25° = -\cos 45°)\right]$

$= 2\sqrt{2}\left[\sin 5° + \sqrt{2} + \cos 35° - \cos 25°\right]$

$= 2\sqrt{2}\left[\sin 5° + \sqrt{2} + 2\sin 30° - \cos 25°\right]$

$= 2\sqrt{2}\left[\sin 5° + \sqrt{2} + 2\sin 30°(-\sin 5°)\right]$

$= 2\sqrt{2} \times \sqrt{2} = 4$

6. (A) → S ; (B) → Q, (C) → P ; (D) → R

$3\left(\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) + \dfrac{4.2\tan\theta}{1 + \tan^2\theta} = 5$

$\Rightarrow \quad 4\tan^2\theta - 4\tan\theta + 1 = 0,$

$\tan\alpha + \tan\beta = 1$

$\tan\alpha \cdot \tan\beta = \dfrac{1}{4}$

$\tan(\alpha+\beta) = \dfrac{4}{3}$ and $\tan(\alpha-\beta) = 0$

7. (A) → P ; (B) → Q, (C) → R ; (D) → P

8. (A) → Q ; (B) → R, (C) → P ; (D) → S

(A) $5\left(\dfrac{1-\cos 2\theta}{2}\right) + 4\left(\dfrac{1+\cos 2\theta}{2}\right) = \dfrac{9}{2} - \dfrac{1}{2}\cos 2\theta$

minimum value $= \dfrac{9}{2} - \dfrac{1}{2} = 4$

(B) $\dfrac{1+\cos\left(\dfrac{2\pi}{3}-2x\right)}{2} - \dfrac{1+\left(\dfrac{2\pi}{3}+2x\right)}{2}$

$= \dfrac{1}{2}\times 2\sin\dfrac{2\pi}{3}\cdot\sin 2x = \dfrac{\sqrt{3}}{2}\cdot\sin 2x$

maximum value $= \dfrac{\sqrt{3}}{2}$

(C) $(\tan\theta - \cot\theta)^2 \ge 0$

$\tan^2\theta + \cot^2\theta \ge 2$

(D) $(3\tan\theta - 2\cot\theta)^2 \ge 0$

$9\tan^2\theta + 4\cot^2\theta \ge 12$

PART - IV INTEGER TYPE

1. (2)

2. (1)

3. (2)

4. (8)

$\dfrac{\tan 3\theta}{\tan\theta} = 4 \quad \Rightarrow \tan^2\theta = \dfrac{1}{11}$

$\dfrac{\sin 3\theta}{\sin\theta} = 3 - 4\sin^2\theta = 3\cdot 4\left(\dfrac{1}{1+\cot^2\theta}\right) = \dfrac{8}{3}$

5. (4)

6. (8)

$\sum_{r=0}^{10}\cos^3\dfrac{r\pi}{3} = \dfrac{1}{4}\sum_{r=0}^{10}\left(3\cos\dfrac{r\pi}{3} + \cos r\pi\right)$

$= \dfrac{1}{4}\left[3\left(\cos 0 + \cos\dfrac{\pi}{3} + \dots + \cos\dfrac{10\pi}{3} + (1 - 1 + \dots - 1 = 1)\right)\right]$

7. (0)

8. (6)

$\sin 10°, \sin 50°, -\sin 70°$ the roots of $8x^3 - 6x + 1 = 0$ .....(i)

let $y = \dfrac{1}{x^2}\left(\text{cosec}^2 10° \text{ etc}\right)$

$\Rightarrow \quad x^2 = \dfrac{1}{y}$

Now from (i), $2x(4x^2 - 3) = -1$

$\Rightarrow \quad 4x^2(4x^2-3)^2 = 1$

$\Rightarrow \quad \dfrac{4}{y}\left(\dfrac{4}{y} - 3\right)^2 = 1$

$\Rightarrow \quad 4(4-3y)^2 = y^3$

$\Rightarrow \quad 4(16 + 9y^2 - 24y) = y^3$

$\Rightarrow \quad y^3 - 36y^2 + 96y - 64 = 0$

roots are $\text{cosec}^2 10°, \text{cosec}^2 50°, \text{cosec}^2 \ne 0°$.

9. (4)

$\text{LHS} = \dfrac{3\cos^2\frac{\pi}{7} - \sin^2\frac{\pi}{7}}{\cos^2\frac{\pi}{7} - \sin^2\frac{\pi}{7}} = \dfrac{4\cos^2\frac{\pi}{7} - 1}{\cos\frac{2\pi}{7}}$

$= \dfrac{1 + 2\cos\frac{2\pi}{7}}{\cos\frac{2\pi}{7}} = \dfrac{2\left[\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7}\right]}{\sin\frac{4\pi}{7}}$

(Multiplying Nr & Dr by $2\sin\dfrac{2\pi}{7}$)

$= \dfrac{4\sin\frac{3\pi}{7}\cdot\cos\frac{\pi}{7}}{\sin\frac{4\pi}{7}} = 4\cos\dfrac{\pi}{7} \quad \left(\because \sin\dfrac{3\pi}{7} = \sin\dfrac{4\pi}{7}\right)$

10. (5)

We have $\tan 19x° = \tan(45° + 96°) = \tan 141°$

$\Rightarrow \quad 19x° = 141° + n(180°)$

$\Rightarrow \quad 19x = 141 + n\cdot 180$

$19x = n\cdot 190 + 190 - 10n - 39$

$$x=(n+1)10-\left(\frac{10n+1}{19}\right)-2$$

x smallest integer x occurs when n = 17

$\therefore$ x = 180 – 19 – 2 = 159

11. (4)

$$S=\frac{1}{1+\left(\tan 1^o\right)^2}+\frac{1}{1+\left(\tan 2^o\right)^2}+\frac{1}{1+\left(\tan 3^o\right)^2}+...+$$

$$\frac{1}{1+\left(\tan 88^o\right)^2}+\frac{1}{1+\left(\tan 89^o\right)^2}$$

reversing the sum

$$S=\frac{1}{1+\left(\cot 1^o\right)^2}+\frac{1}{1+\left(\cot 2^o\right)^2}+....+$$

$$\frac{1}{1+\left(\cot 88^o\right)^2}+\frac{1}{1+\left(\cot 89^o\right)^2}$$

$$2S=\sum_{n=1}^{89}\left(\frac{1}{1+\left(\tan n^o\right)^2}+\frac{1}{1+\left(\cot n^o\right)^2}\right)$$

$$=\sum_{n=1}^{89}\left(\frac{1}{1+\left(\tan n^o\right)^2}+\frac{\left(\tan n^o\right)^2}{1+\left(\tan n^o\right)^2}\right)$$

$$=\sum_{n=1}^{89}1=1+1+....+1=89$$

$\therefore$ S = 44.5

12. 7 Using $\cos 2\alpha=\frac{1-t_2^2}{1+t_2^2}$ where $t_1=\tan\alpha$ and $\cos 2\beta=$ $\frac{1-t_2^2}{1+t_2^2}$ where $t_2=\tan\beta$

we have

$$\left[1003-1002\frac{\left(t-t_1^2\right)}{1+t_1^2}\right]\left[1003-1002\frac{\left(t-t_2^2\right)}{1+t_2^2}\right]$$

$$=\frac{1003\left(1+t_1^2\right)-1002\left(1-t_1^2\right)}{\left(1+t_1^2\right)}\times\frac{1003\left(1+t_2^2\right)-1002\left(1-t_2^2\right)}{\left(1+t_2^2\right)}$$

$$=\frac{\left(1+2005t_1^2\right)\left(1+2005t_2^2\right)}{1+t_1^2+t_2^2+t_1^2t_2^2}\text{ given } t_1t_2=\frac{1}{\sqrt{2005}}$$

$$\Rightarrow\quad t_1^2t_2^2=\frac{1}{2005}$$

$\therefore$ Hence $\frac{1+2005\left(t_1^2+t_2^2\right)+\left(2005\right)^2t_1^2t_2^2}{1+t_1^2+t_2^2+\frac{1}{2005}}$

$$=\frac{1+2005\left(t_1^2+t_2^2\right)+2005}{1+t_1^2+t_2^2+\frac{1}{2005}}$$

$$=\frac{2005\left[\frac{1}{2005}+t_1^2+t_2^2+1\right]}{\left[1+t_1^2+t_2^2+\frac{1}{2005}\right]}=2005$$

13. Since $\sqrt{x^2}=|x|$ for real x, we can now simplify the function as

$$f(x)=\frac{\sin x}{|\sin x|}+\frac{\cos x}{|\cos x|}+\frac{\tan x}{|\tan x|}+\frac{\cot x}{|\cot x|}$$

If $x\in Q_1$, f(x) = 1 + 1 + 1 + 1 = 4

If $x\in Q_2$, f(x) = 1 – 1 – 1 – 1 = 4

If $x\in Q_3$, f(x) = 1 – 1 + 1 + 1 = 0

If $x\in Q_4$, f(x) = 1 – 1 – 1 – 1 = –2

$\therefore$ $f_{min}=-2$

14. (0009)

$$E=\frac{\sin\theta+\sin 2\theta+\sin 3\theta\sin 6\theta+\sin 4\theta\sin 13\theta}{\sin\theta\cos 2\theta+\sin 3\theta\cos 6\theta+\sin 4\theta\cos 13\theta}$$

$$=\frac{2\sin\theta\sin 2\theta+2\sin 3\theta\sin 6\theta+2\sin 4\theta\sin 13\theta}{2\sin\theta\cos 2\theta+2\sin 3\theta\cos 6\theta+2\sin 4\theta\cos 13\theta}$$

$$=\frac{\left(\cos\theta-\cos 3\theta\right)+\left(\cos 3\theta-\cos 9\theta\right)+\left(\cos 9\theta-\cos 17\theta\right)}{\left(\sin 3\theta-\sin\theta\right)+\left(\sin 9\theta-\sin 3\theta\right)+\left(\sin 17\theta-\sin 9\theta\right)}$$

$$=\frac{\cos\theta-\cos 17\theta}{\sin 17\theta-\sin\theta}=\frac{2\sin 9\theta\sin 8\theta}{2\cos 9\theta\sin 8\theta}=\tan(9\theta)$$

= tan (kθ)

Hence, k = 9.

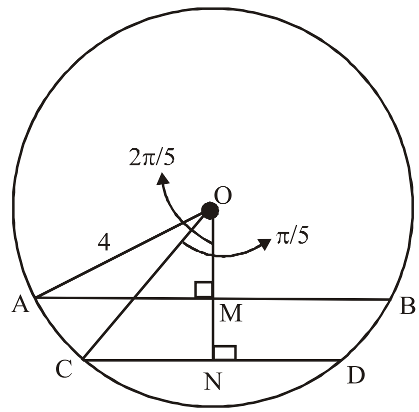
15. (1)

$$\frac{\sin^2 34^o-\sin^2 11^o}{\sin 34^o\cos 34^o-\sin 11^o\cos 11^o}=\frac{2[\sin 45^o\cdot\sin 23^o]}{\sin 68^o-\sin 22^o}$$

$$= \frac{2\sin 45^o \sin 23^o}{2\cos 45^o \sin 23^o} = \tan 45^o = 1$$

16. (2)

$OM = 4\cos\frac{2\pi}{5}$



$ON = 4\cos\frac{2\pi}{5}$

$$MN = ON - OM = 4\left(\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}\right)$$

$$= 4\left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right) = 4\left(\frac{1}{2}\right) = 2$$

17. (0007)

Given $N = \prod_{r=1}^{7} \sec\left(\frac{r\pi}{15}\right)$

$$= \frac{1}{\cos\frac{\pi}{15}\cdot\cos\frac{2\pi}{15}\cdot\cos\frac{3\pi}{15}\cdot\cos\frac{4\pi}{15}\cdot\cos\frac{5\pi}{15}\cdot\cos\frac{6\pi}{15}\cdot\cos\frac{7\pi}{15}}$$

$$= \frac{1}{\cos 12^o\cdot\cos 24^o\cdot\cos 36^o\cdot\cos 48^o\cdot\cos 60^o\cdot\cos 72^o\cdot\cos 84^o}$$

$$= \frac{2}{\left(\cos 12^o\cdot\cos 48^o \cos 72^o\right)\left(\cos 24^o\cdot\cos 36^o\cdot\cos 84\right)}$$

$$= \frac{1}{\left(\frac{1}{4}\cos\left(3\times 12^o\right)\right)\left(\frac{1}{4}\cos\left(3\times 24^o\right)\right)}$$

[As, $\cos\theta\cdot\cos(60^o-\theta)\cdot\cos(60^o+\theta) = 1/4\cos 3\theta$]

$$= \frac{32}{\cos 36^o \times \cos 72^o} = \frac{32}{\left(\frac{\sqrt{5}+1}{4}\right)\times\left(\frac{\sqrt{5}-1}{4}\right)}$$

$\therefore$ $N = 32\times 4 = 128 = 2^7$. Hence, $\log_2 N = 7$.

18. (1)

$$\prod_{k=0}^{5} 2\cos\left(2^k A\right) = 2^6 \cos A\cos 2A\cos 2^2 A.....\cos 2^5 A$$

$$= \frac{\sin\left(2^6 A\right)}{\sin A} = \frac{\sin\left(\frac{64\pi}{65}\right)}{\sin\left(\frac{\pi}{65}\right)} = 1$$

19. (9)

$$p + q = \sum_{n=1}^{9}\sin^2\left(\frac{n\pi}{24}\right) + \sum_{n=1}^{9}\cos^2\left(\frac{n\pi}{24}\right),$$

$$= \sum_{n=1}^{9}\left[\sin^2\left(\frac{n\pi}{24}\right) + \cos^2\left(\frac{n\pi}{24}\right)\right] = \sum_{n=1}^{9} 1 = 9$$

20. (1)

$T_r = \tan r^\circ (\tan (r+1)^\circ + 1 - 1$

$$T_r = \frac{\sin r^o \sin\left(r+1\right)^o \cos r^o \cos\left(r+1\right)^o}{\cos r^o \cos\left(r+1\right)^o} - 1$$

$$T_r = \frac{\cos 1^o}{\cos r^o \cos\left(r+1\right)^o} - 1$$

$$T_r = \frac{\cos 1^o}{\sin 1^o}\left(\frac{\sin\left(r+1\right)^o - r^o}{\cos r^o \cos r\left(r+1\right)^o}\right) - 1$$

$T_r = \cot 1^\circ (\tan(r+1)^\circ - \tan r^\circ) - 1$

$T_1 + T_2 + ... + T_{18} = \cot^2 1^\circ - 89.$

21. (5)

$x^2 + y^2 = 4$

Let $x = 2\sin\theta$, $y = 2\cos\theta$ $\Rightarrow$ $E = (1-2x^2)^2 + 4x^2y^2$

$E = 25 - 24\cos 2\theta$ $\quad\therefore\quad$ $m = 1$ and $M = 49$

22. (0019)

Let $\frac{\pi}{15} = \theta \Rightarrow 15\theta = \pi$ and $\theta = 12^o$

$(\sin 2\theta + \sin 4\theta) + (\sin 7\theta - \sin\theta)$

$\Rightarrow$ $2\sin 3\theta\cos\theta + 2\cos 4\theta\sin 3\theta$

$2\sin 3\theta\,[\cos 4\theta + \cos\theta]$

$\Rightarrow$ $2\sin 36^\circ\,[\cos 48^\circ + \cos 12^\circ]$

$2\sin 36^\circ\,[2\cos 30^\circ\cos 18^\circ]$

$\Rightarrow$ $\sqrt{3}\cdot 2\sin 36^o\cos 18^o \Rightarrow \sqrt{3}[\sin 54^o + \sin 18^o]$

$$\Rightarrow \sqrt{3}\left[\frac{\sqrt{5}+1}{4}+\frac{\sqrt{5}-1}{4}\right] \Rightarrow \sqrt{3}\left[\frac{\sqrt{5}}{2}\right] \Rightarrow \sqrt{\frac{15}{4}}=\sqrt{\frac{a}{b}}$$

$\Rightarrow (a+b)=19.$

23. (3)

$OM = 2\cdot\cos\frac{\pi}{K}; ON = 2\cos\frac{\pi}{2K}$ Clearly,

Now, $OM + ON = \sqrt{3}+1$(Given)

$$\Rightarrow 2\cdot\cos\frac{\pi}{K}+2\cdot\cos\frac{\pi}{2K}=\left(\sqrt{3}+1\right)$$

$$\therefore \cos\frac{\pi}{K}+\cos\frac{\pi}{2K}=\left(\frac{\sqrt{3}+1}{2}\right)$$

Let $\frac{\pi}{2K}=\theta$, so $\frac{\pi}{K}=2\theta$

$\therefore$ We have $(\cos 2\theta + \cos\theta) = \frac{\sqrt{3}+1}{2}$

$$\Rightarrow 2\cos^2\theta-1+\cos\theta=\left(\frac{\sqrt{3}+1}{2}\right)$$

$$\Rightarrow 2\cos^2\theta+\cos\theta-\left(\frac{3+\sqrt{3}}{2}\right)=0$$

$$\therefore \cos\theta=\frac{-1\pm\sqrt{1+4\left(3+\sqrt{3}\right)}}{4}$$

$$\Rightarrow \cos\theta=\frac{-1\pm\sqrt{\left(2\sqrt{3}+1\right)^2}}{4}=\frac{-1\pm\left(2\sqrt{3}+1\right)}{4}$$

$$\Rightarrow \cos\theta=\frac{\sqrt{3}}{2},\frac{-1-\sqrt{3}}{2}$$

As, $\cos\theta = \frac{-1-\sqrt{3}}{2}$ is not possible (Rejected)

$$\therefore \cos\theta=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{6}=\frac{p}{2K} \Rightarrow K=3$$

$\therefore [K]=3$

PART - V SUBJECTIVE TYPE

1. $\frac{\sin x-\cos x+1}{\sin x+\cos x-1}$

$$=\frac{(\sin^2 x)-(\cos x-1)^2}{(\sin x+\cos x-1)^2}$$

$$=\frac{2\cos x-2\cos 2x}{2+2\sin x\cos x-2\cos x-2\sin x}$$

$$=\frac{2\cos x(1-\cos x)}{2(1-\sin x)(1-\cos x)}$$

$$=\frac{\cos x}{1-\sin x}=\frac{(1+\sin x)}{\cos x}=\sec x+\tan x$$

2. Given, $2\cos x+\sin x=1$

or $4\cos^2 x=(1-\sin x)^2$

or $4-4\sin^2 x=1+\sin^2 x-2\sin x$

or $5\sin^2 x-2\sin x-3=0$

or $(\sin x-1)(5\sin x+3)=0$

or $\sin x=1, \sin x=-\frac{3}{5}$

For $\sin x = 1$, we have

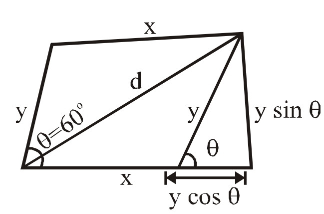
$7\cos x+6\sin x=0+6=6$

and for $\sin x=\frac{-3}{5}$, we have

$$7\cos x+6\sin x=7\left(\frac{1+\frac{3}{5}}{2}\right)-\frac{6\times 3}{5}$$

$$=\frac{28-18}{5}=2$$

3.



Perimeter,

$2(x+y)=p$ or $x+y=\frac{p}{2}$ .....(i)

Also,

$d^2=(x+y\cos\theta)^2+y^2\sin^2\theta$

$=x^2+y^2+2xy\cos\theta$

$=x^2+y^2+xy$ $(\theta=60^o)$

$=(x+y)^2-xy$

$$= \frac{p^2}{4} - xy \qquad \text{[using (i)]}$$

or $xy = \frac{p^2}{4} - d^2$

Now,

Area of parallelogram

$$= xy \sin 60^o = \left(\frac{p^2}{4} - d^2\right)\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}(p^2 - 4d^2)$$

4. $$\frac{u_5 - u_7}{u_3 - u_5} = \frac{(\sin^5\theta + \cos^5\theta) - (\sin^7\theta + \cos^7\theta)}{(\sin^3\theta + \cos^3\theta) - (\sin^5\theta + \cos^5\theta)}$$

$$= \frac{\sin^5\theta(1\sin^2\theta) + \cos^5\theta(1-\cos^2\theta)}{\sin^3\theta(1-\sin^2\theta) + \cos^3\theta(1-\cos^2\theta)}$$

$$= \frac{\sin^2\theta\cos^2\theta[\sin^3\theta + \cos^3\theta]}{\sin^2\theta\cos^2\theta[\sin\theta + \cos\theta]} = \frac{u_3}{u_1}$$

5. $A = \sin x + \cos x$

$\therefore$ $A^2 = 1 + 2\sin x \cos x$

Now, $\sin^4 + \cos^4 x$

$= (\sin^2 x + \cos^2 x) - 2\sin^2 x$

$= 1 - 2\sin^2 \cos^2 x$

$$= 1 - 2\left(\frac{A^2 - 1}{2}\right)^2$$

$$= 1 - \frac{(A^2-1)^2}{2}$$

$$= \frac{2 - (A^4 - 2A^2 + 1)}{2}$$

$$= \frac{1 + 2A^2 - A^4}{2}$$

$$= \frac{1}{2} + A^2 - \frac{1}{2}A^4$$

6. $$\frac{\sec^4\theta}{a} + \frac{\tan^4\theta}{b} = \frac{1}{a+b} = \frac{\sec^2\theta - \tan^2\theta}{a+b}$$

$$\Rightarrow \frac{\sec^2\theta}{a(a+b)}[(a+b)\sec^2\theta - a] +$$

$$\frac{\tan^2\theta}{b(a+b)}[(a+b)\tan^2\theta + b] = 0$$

$\Rightarrow$ $a\tan^2\theta + b\sec^2\theta = 0$

$\Rightarrow$ $\sin^2\theta = -\frac{b}{a}$

Since $\sin^2\theta \le 1$

$\Rightarrow$ $\left|\frac{b}{a}\right| \le 1 \text{or} \ |b| \le |a|$

7. $A + B + C = \pi$

or $\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

or $\cot\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\left(\frac{\pi}{2} - \frac{C}{2}\right)$

or $$\frac{\cos\frac{A}{2}\cot\frac{B}{2} - 1}{\cot\frac{A}{2} + \cot\frac{B}{2}} = \tan\frac{C}{2} = \frac{1}{\cot\frac{C}{2}}$$

or $\cos\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

But $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P. ....(i)

$\therefore$ $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.

So, $\cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2}$

Hence, Eq. (i) becomes

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = 3\cot\frac{B}{2} \quad \Rightarrow \quad \cot\frac{A}{2}\cot\frac{C}{2} = 3$$

Thus, G.M. of $\cot\frac{A}{2}$ and $\cot\frac{C}{2}$ is

$$\sqrt{\cot\frac{A}{2}\cot\frac{C}{2}} = \sqrt{3}$$

and A.M. of $\cot\frac{A}{2}$ and $\cot\frac{C}{2}$ is

$$\frac{\cot\frac{A}{2} + \cot\frac{C}{2}}{2} = \cot\frac{B}{2}$$

But A.M. ≥ G.M. Thus,

$$\cot\frac{B}{2} \ge \sqrt{3}$$

Therefore, the minimum value of cot B/2 is $\sqrt{3}$.

8. $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

$$= \frac{1}{\sin\theta}\frac{\sin\theta/2}{\sin\theta/2}$$

$$= \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\sin\theta\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\sin\theta\sin\frac{\theta}{2}}$$

$\therefore$ $\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \cot\theta$

Similarly, cosec $2\theta = \cot\theta - \cot 2\theta$

cosec $4\theta = \cot 2\theta - \cot 4\theta$

cosec $4\theta = \cot 2\theta - \cot 2^{n-1}\theta - 2^{n-1}\theta$

$\therefore$ Sum $= \cot\dfrac{\theta}{2} - \cot 2^{n-1}\theta$

9. Here, we have $\tan 6\theta = p/q$

or $\dfrac{\sin 6\theta}{\cos 6\theta} = \dfrac{p}{q}$

or $\dfrac{p}{\sin 6\theta} = \dfrac{q}{\cos 6\theta} = \dfrac{\sqrt{p^2+q^2}}{\sqrt{1}}$

Now $y = \dfrac{1}{2}(p\,\text{cosec}\,2\theta - q\sec 2\theta)$

$= \dfrac{1}{2}\left(\dfrac{p}{\sin 2\theta} - \dfrac{q}{\cos 2\theta}\right)$

$= \dfrac{1}{2}\left[\dfrac{p\cos 2\theta - q\sin 2\theta}{\sin 2\theta\cos 2\theta}\right]$

$= \left[\dfrac{2k\sin 6\theta\cos 2\theta - 2k\cos 6\theta\sin 2\theta}{4\sin 2\theta\cos 2\theta}\right]$

$= k\dfrac{\sin(6\theta - 2\theta)}{\sin 4\theta} = k = \sqrt{p^2+q^2}$

10. $\sin\theta + \cos\alpha(\tan\alpha + \cot\alpha) + (\sec\alpha + \text{cosec}\,\alpha) = 7$

or $(\sin\alpha + \cos\alpha) + \dfrac{1}{\sin\alpha\cos\alpha} + \dfrac{\sin\alpha + \cos\alpha}{\sin\alpha\cos\alpha} = 7$

or $(\sin\alpha + \cos\alpha)\left(1 + \dfrac{1}{\sin\alpha\cos\alpha}\right)$

$= 7 - \dfrac{1}{\sin\alpha\cos\alpha}$

or $(1 + \sin 2\alpha)\left(1 + \dfrac{4}{\sin 2\alpha} + \dfrac{4}{\sin^2 2\alpha}\right)$

$= 49 - \dfrac{28}{\sin 2\alpha} + \dfrac{4}{\sin^2 2\alpha}$

Let $\sin 2\alpha = x$. Then,

$(1+x)\left(1 + \dfrac{4}{x} + \dfrac{4}{x^2}\right) = 49 - \dfrac{28}{x} + \dfrac{4}{x^2}$

or $(1+x)(x^2+4x+4) = 49x^2 - 28x + 4$

or $x^3 - 44x^2 + 36x = 0$

or $x^2 - 44x + 36 = 0$ (as $x = \sin 2\alpha \neq 0$)

11. $A + B + C = \pi$

$\Rightarrow B + C = \dfrac{3\pi}{4} \Rightarrow 0 < B, C < \dfrac{3\pi}{4}$

Also $\tan B\tan C = p$

or $\dfrac{\sin B\sin C}{\cos B\cos C} = \dfrac{p}{1}$

or $\dfrac{\cos B\cos C - \sin B\sin C}{\cos B\cos C + \sin B\sin C} = \dfrac{1-p}{1+p}$

or $\dfrac{\cos(B+C)}{\cos(B-C)} = \dfrac{1-p}{1+p}$

or $\dfrac{1+p}{\sqrt{2}(p-1)} = \cos(B-C)$

Since B or C can vary from 0 to $3\pi/4$, we get

$0 < B - C < \dfrac{3\pi}{4}$ or $-\dfrac{1}{\sqrt{2}} < \cos(B-C) \le 1$

Equation (i) will now lead to $-\dfrac{1}{\sqrt{2}} < \dfrac{p+1}{\sqrt{2}(p-1)} \le 1$

For $0 < 1 + \dfrac{p+1}{p-1}$ or $\dfrac{2p}{(p-1)} > 0$

$\therefore$ $p < 0$ or $p > 1$ ....(ii)

Also $\dfrac{p+1-\sqrt{2}(p-1)}{\sqrt{2}(p-1)} \le 0$

or $\dfrac{\left(p - \left(\sqrt{2}+1\right)^2\right)}{(p-1)} \le 0$

$\therefore$ $p < 1$ or $\ge \left(\sqrt{2}+1\right)^2$ ....(iii)

Combining Eqs. (ii) and (iii), we get $p < 0$ or $p \ge \left(\sqrt{2}+1\right)^2$.

12. L.H.S. contains x, 3x, 9x, and 27x, whereas R.H.S contains 27x and x only. So, we will manipulate terms as shown below :

R. H. S. $= \dfrac{1}{2}[\tan 27x - \tan x]$

$= \dfrac{1}{2}[(\tan 27x - \tan 9x) + (\tan 9x - \tan 3x) + (\tan 3x - \tan x)]$

$= \dfrac{1}{2}\left(\dfrac{\sin 27x}{\cos 27x} - \dfrac{\sin 9x}{\cos 9x}\right)$

$+ \left(\dfrac{\sin 9x}{\cos 9x} - \dfrac{\sin 3x}{\cos 3x}\right) + \left(\dfrac{\sin 3x}{\cos 3x} - \dfrac{\sin x}{\cos x}\right)\Bigg]$

$= \dfrac{1}{2}\left[\dfrac{\sin(27x-9x)}{\cos 27x\cos 9x} + \dfrac{\sin(9x-3x)}{\cos 9x\cos 3x} + \dfrac{\sin(3x-x)}{\cos 3x\cos x}\right]$

$= \dfrac{1}{2}\left[\dfrac{\sin 18x}{\cos 27x\cos 9x} + \dfrac{\sin 6x}{\cos 9x\cos 3x} + \dfrac{\sin 2x}{\cos 3x\cos x}\right]$

$$= \frac{1}{2}\left[\frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x} + \frac{23x\cos 3x}{\cos 9x\cos 3x} + \frac{2\sin x\cos x}{\cos 3x\cos x}\right]$$

$$= \frac{\sin 9x}{\cos 27x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin x}{\cos 3x}$$

$$= \frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = \text{L.H.S.}$$

13. We have to prove that

$$\frac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ .....$$

$(2\cos 2^{n-1}\theta - 1)$

or $\quad 2\cos 2^n\theta + 1$

$= [(2\cos\theta + 1)(2\cos\theta - 1)](2\cos 2\theta - 1)$

$(2\cos 2^2\theta - 1)\ .....\ (2\cos 2^{n-1}\theta - 1)$

Now

$[(2\cos\theta + 1)(2\cos\theta - 1)](2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ .....$

$(2\cos 2^{n-1}\theta - 1)$

$= (4\cos^2\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ .....$

$(2\cos 2^{n-1}\theta - 1)$

$= (2\cos 2\theta + 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\ ......$

$(2\cos 2^{n-1}\theta - 1)$ [using $\cos 2\theta = 2\cos^2\theta - 1$]

$= (4\cos^2 2\theta - 1)(2\cos 2^2\theta - 1)\ .....\ (2\cos 2^{n-1}\theta - 1)$

$= (2\cos 2^2\theta + 1)(2\cos 2^2\theta - 1)\ .....\ (2\cos 2^{n-1}\theta - 1)$

$= (4\cos^2 2^2\theta - 1)(2\cos 2^3\theta - 1)\ .....\ (2\cos 2^{n-1}\theta - 1)$

......

......

......

$= (2\cos 2^{n-1}\theta + 1)(2\cos 2^{n-1}\theta - 1)$

$= 4\cos^2 2^{n-1}\theta - 1$

$= 2\cos 2^n\theta + 1$

14. We have to prove that

$$\frac{\tan 2^n\theta}{\tan\theta} = (1 + \sec 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$$

or $\quad \tan 2^n\theta = \tan\theta(1 + \sec 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)$

$\cdots(1 + \sec 2^n\theta)$

Now,

$\tan\theta(1 + \sec 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$

$$= \tan\theta\left(\frac{1 + \cos 2\theta}{\cos 2\theta}\right)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$$

$$= \frac{\sin\theta}{\cos\theta}\left(\frac{2\cos^2\theta}{\cos 2\theta}\right)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$$

$= (\tan 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$

$$= (\tan 2\theta)\left(\frac{1 + \cos 2^2\theta}{\cos 2^2\theta}\right)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$$

$$= (\tan 2\theta)\left(\frac{2\cos^2 2\theta}{\cos 2^2\theta}\right)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$$

$= (\tan 2^2\theta)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$

$\vdots$

$= \tan 2^{n-1}\theta(1 + \sec 2^n\theta)$

$$= \tan 2^{n-1}\theta\left(\frac{1 + \cos 2^n\theta}{\cos 2^n\theta}\right)$$

$$= \tan 2^{n-1}\theta\left(\frac{2\cos^2 2^{n-1}\theta}{\cos 2^n\theta}\right)$$

$= \tan 2^n\theta$

15. $$X = \sin\left(\theta + \frac{7\pi}{12}\right) + \sin\left(\theta - \frac{\pi}{12}\right) + \sin\left(\theta + \frac{3\pi}{12}\right)$$

$$= 2\sin\left(\theta + \frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\theta + \frac{3\pi}{12}\right)$$

$$= \sin\left(\theta + \frac{\pi}{4}\right) + \sin\left(\theta + \frac{\pi}{4}\right)$$

$$= 2\sin\left(\theta + \frac{\pi}{4}\right)$$

$$Y = \cos\left(\theta + \frac{7\pi}{12}\right) + \cos\left(\theta - \frac{\pi}{12}\right) + \cos\left(\theta + \frac{3\pi}{12}\right)$$

$$= 2\cos\left(\theta + \frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\theta + \frac{3\pi}{12}\right)$$

$$= 2\cos\left(\theta + \frac{\pi}{4}\right)$$

$$\Rightarrow \quad \frac{X^2 - Y^2}{XY} = \frac{4\sin^2\left(\theta + \frac{\pi}{4}\right) - 4\cos^2\left(\theta + \frac{\pi}{4}\right)}{4\sin\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta + \frac{\pi}{4}\right)}$$

$$= -\frac{2\cos 2\left(\theta + \frac{\pi}{4}\right)}{\sin\left(2\theta + \frac{\pi}{2}\right)} = \frac{2\sin 2\theta}{\cos 2\theta} = 2\tan 2\theta$$

16. We have $\tan(A - B) + \tan(B - C) + \tan(C - A) = 0$

Let $\tan A = p$, $\tan B = q$ and $\tan C = r$

$$\Rightarrow \quad \frac{p - q}{1 + pq} + \frac{q - r}{1 + qr} + \frac{r - p}{1 + rp} = 0$$

$\Rightarrow \quad (p - q)(1 + qr)(1 + rp) + (q - r)(1 + pq)(1 + rp) + (r - p)$

$(1 + pq)(1 + qr) = 0$

$\Rightarrow \quad (p - q)(1 + r(p + q) + pqr^2) + (q - r)(1 + p(q + r) + qrp^2)$

$+ (r - p)(1 + q(r + p) + rpq^2) = 0$

$\Rightarrow \quad (r(p^2 - q^2) + (p - q)pqr^2) + (p(q^2 - r^2) + (q - r)$

$(qrp^2) + (q(r^2 - p^2) + (r - p)rpq^2) = 0$

$\Rightarrow$ $r(p^2 - q^2) + p(q^2 - r^2) + q(r^2 - p^2) = 0$

$\Rightarrow$ $rp^2 - rq^2 + pq^2 - pr^2 + qr^2 - qp^2 + pqr - pqr = 0$

$\Rightarrow$ $(p - q)(q - r)(r - p) = 0$

$\Rightarrow$ $p = q, q = r$ or $r = p$

Thus, triangle is isosceles.

17. $A + B = 90°$

$\Rightarrow$ $B = 90° - A$

$\therefore$ $\tan B = \dfrac{1}{\tan A}$

Now, given equation becomes

$$\left(\tan A + \frac{1}{\tan A}\right) + \left(\tan^2 A + \frac{1}{\tan^2 A}\right) + \left(\tan^3 A + \frac{1}{\tan^3 A}\right) = 70$$

$$\Rightarrow \left(\tan A + \frac{1}{\tan A}\right) + \left(\tan A + \frac{1}{\tan A}\right)^2 - 2$$

$$+ \left(\tan A + \frac{1}{\tan A}\right)^3 - 3\left(\tan A + \frac{1}{\tan A}\right) = 70$$

Let $\left(\tan A + \dfrac{1}{\tan A}\right) = t$

So, $t + t^2 - 2 + t^3 - 3t = 70$

$\Rightarrow$ $t^3 + t^2 - 2t - 72 = 0$

Using hit and trial, we get $t = 4$.

$\therefore$ $\tan A + \dfrac{1}{\tan A} = 4$

$\Rightarrow$ $\dfrac{2(\tan^2 A + 1)}{2\tan A} = 4$

$\Rightarrow$ $2\left(\dfrac{1}{\sin 2A}\right) = 4$

$\Rightarrow$ $\sin 2A = 1/2$

$\therefore$ $2A = 30°$

$\therefore$ $A = 15°$

$\therefore$ $B = 75°$

EXERCISE - IV

1. (C)

$$\cot\frac{(x+y)}{2}=\frac{\cos\left(\frac{x+y}{2}\right)}{\sin\left(\frac{x+y}{2}\right)} \quad .....(i)$$

$\cos x+\cos y+\cos\alpha=0$

$$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)+\cos\alpha=0 \quad .....(ii)$$

$\sin x+\sin y+\sin\alpha=0$

$$2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)+\sin\alpha=0 \quad .....(iii)$$

Equation (i), (ii), (iii)

$$\cot\left(\frac{x+y}{2}\right)=\frac{-\left(\dfrac{\cos\alpha}{2\cos\left(\frac{x-y}{2}\right)}\right)}{-\left(\dfrac{\sin\alpha}{2\cos\left(\frac{x-y}{2}\right)}\right)}=\cot\alpha$$

2. (0)

$\cos 1^\circ\cos 2^\circ.\cos 3^\circ \,.......\, \cos 179^\circ=$

$\cos 1^\circ\cos 2^\circ.\cos 3^\circ \,.......\, \cos 90^\circ \,......\, \cos 179^\circ=0$

$\therefore\quad [\cos 90^\circ=0]$

3. (A)

Given that

$$\sin\alpha+\sin\beta=-\frac{21}{65} \quad .....(i)$$

$$\text{and } \cos\alpha+\cos\beta=-\frac{27}{65} \quad .....(ii)$$

On squaring and adding Equ. (i) and (ii), we get

$\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta$

$\qquad +\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta$

$$=\left(-\frac{21}{65}\right)^2+\left(-\frac{27}{65}\right)^2$$

$$\Rightarrow\ 2+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=\frac{441}{4225}+\frac{729}{4225}$$

$$\Rightarrow\ 2[1+\cos(\alpha-\beta)]=\frac{1170}{4225}$$

$$\Rightarrow\ \cos^2\left(\frac{\alpha-\beta}{2}\right)=\frac{1170}{4\times 4225}$$

$$\Rightarrow\ \cos^2\left(\frac{\alpha-\beta}{2}\right)=\frac{9}{130}$$

$$\Rightarrow\ \cos\left(\frac{\alpha-\beta}{2}\right)=-\frac{3}{\sqrt{130}}$$

$$\left(\because \pi<\alpha-\beta<3\pi\Rightarrow\frac{\pi}{2}<\frac{\alpha-\beta}{2}<\frac{3\pi}{2}\right)$$

4. (B)

$$\cos(\alpha+\beta)=\frac{4}{5}\quad\Rightarrow\quad \alpha+\beta\in 1^{st}\text{ quadrant}$$

$$\text{and } \sin(\alpha-\beta)=\frac{5}{13}\quad\Rightarrow\quad \alpha-\beta\in 1^{st}\text{ quadrant}$$

$\Rightarrow\quad 2\alpha=(\alpha+\beta)+(\alpha-\beta)$

$$\therefore\quad \tan 2\alpha=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}=\frac{56}{33}$$

5. (B)

$A=\sin^2 x+\cos^4 x$

$\quad =\sin^2 x+(1-\sin^2 x)^2$

$\quad =\sin^4 x-\sin^2 x+1$

$\quad =(\sin^2 x-1/2)^2+3/4$

6. (A)

$3\sin P+4\cos Q=6 \quad .....(i)$

$4\sin Q+3\cos P=1 \quad .....(ii)$

From (i) and (ii) $\angle P$ is obtuse.

$(3\sin P+4\cos Q)^2+(4\sin Q+3\cos P)^2=37$

$\Rightarrow\quad 9+16+24(\sin P\cos Q+\cos P\sin Q)=37$

$\Rightarrow\quad 24\sin(P+Q)=12$

$$\Rightarrow\quad \sin(P+Q)=\frac{1}{2}\quad\Rightarrow\quad P+Q=\frac{5\pi}{6}\quad\Rightarrow\quad R=\frac{\pi}{6}$$

7. (B, C)

All are infinite geometric progression with common ratio < 1.

$$x=\frac{1}{1-\cos^2\phi}=\frac{1}{\sin^2\phi},\ y=\frac{1}{1-\sin^2\phi}=\frac{1}{\cos^2\phi},$$

$$z=\frac{1}{1-\cos^2\phi\sin^2\phi}$$

$$\text{Now, } xy+z=\frac{1}{\sin^2\phi+\cos^2\phi}+\frac{1}{1-\sin^2\phi\cos^2\phi}$$

$$=\frac{1}{\sin^2\phi\cos^2\phi(1-\sin^2\phi\cos^2\phi)}$$

or $\quad xy+z=xyz \quad .....(i)$

$$\text{Clearly, } x+y=\frac{\sin^2\phi+\cos^2\phi}{\sin^2\phi\cos^2\phi}$$

$\therefore\quad x+y+z=xyz \qquad$ [Using Equ. (i)]

8. (A) $\because \quad \pi/18 = 10°$

We know that $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

By complementary rule

$\Rightarrow \quad \sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$

$\Rightarrow \quad \frac{1}{2}\sin 10° \sin 50° \sin 70° = \frac{1}{16}$

$\Rightarrow \quad \sin 10° \sin 50° \sin 70° = \frac{1}{8}$

So, $k = \frac{1}{8}$

9. (B)

Given $A > 0$ and $B > 0$; $A + B = \frac{\pi}{3}$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$\tan \frac{\pi}{3} = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$\sqrt{3} = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$\sqrt{3}(1 - \tan A . \tan B) = \tan A + \tan B$ .....(i)

since AM $\geq$ GM

$\frac{\tan A + \tan B}{2} \geq \sqrt{\tan A \cdot \tan B}$

$\Rightarrow \quad \tan A + \tan B \geq 2\sqrt{\tan A \cdot \tan B}$ .....(ii)

From Equ. (i) and (ii)

$\sqrt{3}(1 - \tan A . \tan B) \geq 2\sqrt{\tan A \tan B}$

$\Rightarrow \quad 3(1 - \tan A . \tan B)^2 \geq 4 \tan A . \tan B$

let $\quad y = \tan A . \tan B$

$3(1 - y)^2 \geq 4y$

$\Rightarrow \quad 3y^2 - 6y + 3 \geq 4y$

$\Rightarrow \quad 3y^2 - 10y + 3 \geq 0$

$\Rightarrow \quad 3y^2 - 9y - y + 3 \geq 0$

$\Rightarrow \quad 3y(y-3) - 1(y-3) \geq 0$

$\Rightarrow \quad (3y-1)(y-3) \geq 0$

From wave curve method

from wave curve method

$\Rightarrow \quad y \leq \frac{1}{3} \quad$ or $\quad y \geq 3$

Since $0 < A < \frac{\pi}{3}$ and $0 < B < \frac{\pi}{3}$;

$0 < \tan A < \sqrt{3}$ and $0 < \tan B < \sqrt{3}$

$\Rightarrow \quad \tan A . \tan B < \sqrt{3} . \sqrt{3} \quad \Rightarrow \quad y < 3$

$\Rightarrow \quad y = \leq \frac{1}{3}$

Thus the maximum value of tan A . tan B is 1/3

10. (B)

$3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi - \alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$

$= 3\left[\cos^4\alpha + \sin^4\alpha\right] - 2\left[\cos^6\alpha + \sin^6\alpha\right]$

$= 3\left[(\cos^2\alpha + \sin^2\alpha)^2 - 2\cos^2\alpha \cdot \sin^2\alpha\right]$

$= 2[(\cos^2\alpha + \sin^2\alpha)^3 - 3\cos^2\alpha \cdot \sin^2\alpha(\cos^2\alpha + \cos^2\alpha)]$

$= 3[1 - 2\sin^2\alpha \cdot \cos^2\alpha] - 2[1 - 3\sin^2\alpha \cdot \cos^2\alpha]$

$= 3 - 6\sin^2\alpha \cdot \sin^2\alpha - 2 + 6\sin^2\alpha \cdot \cos^2\alpha$

$= 1$

11. (C)

$3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4(\sin^6\theta + \cos^6\theta)$

$= 3[\sin^4\theta + \sin^4\theta - \sin^3\theta\cos\theta$

$+ 6\sin^2\cos^2\theta - 4\sin\theta\cos^3\theta]$

$+ 6[1 + 2\sin\theta\cos\theta]$

$+ 4[\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta]$

$= 7[\sin^4\theta + \cos^4\theta] + 14\sin^2\theta\cos^2\theta$

$- 12\sin\theta\cos\theta + 6 + 12\sin\theta\cos\theta$

$= 7(\sin^2\theta + \cos^2\theta)^2 + 6 = 13$

12. (B)

Given, $\sec^2\theta = \frac{4xy}{(x+y)^2}$

Now, $\sec^2\theta \geq 1 \quad \Rightarrow \quad \frac{4xy}{(x+y)^2} \geq 1$

or $\quad (x+y)^2 \leq 4xy$

or $\quad (x+y)^2 - 4xy \leq 0$

or $\quad (x-y)^2 \leq 0$

But for real values of x and y,

$(x-y)^2 \geq 0 \quad$ or $\quad (x-y)^2 = 0$

$\therefore \quad x = y$

Also $x + y \neq 0 \quad \Rightarrow \quad x \neq 0. y \neq 0$

13. (A)

$f(x) = \cos x + \cos(\sqrt{2}x)$

$f'(x) = -\sin x - \sqrt{2}\sin(\sqrt{2}x)$

$f'(x) = 0$

$-\sin x - \sqrt{2}\sin(\sqrt{2}x)$

$\sin x = -\sqrt{2}\sin(\sqrt{2}x)$

True only for $x = 0$

$f''(x) = -\cos x - 2\cos(\sqrt{2}x)$

$f''(0) = -1 - 2 = -3 < 0$

$f(x)$ maximum at $x = 0$

14. (C)

$\frac{1}{2} . 2\sin 15° \cos 15° = \frac{1}{2}\sin 30° = \frac{1}{4}$ i.e. rational

15. (B)

$$\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta \qquad \text{(given)}$$

now put $\theta = 0$, we get $0 = b_0$

$$\therefore \quad \sin n\theta = \sum_{r=1}^{n} b_r \sin^r \theta \text{ is true}$$

$$\Rightarrow \quad \frac{\sin n\theta}{\sin\theta} = \sum_{r=1}^{n} b_r (\sin\theta)^{r-1}$$

Taking limit as $\theta \to 0$

$$\Rightarrow \quad \lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta} = \lim_{\theta\to 0}\sum_{r=1}^{n} b_r (\sin\theta)^{r-1}$$

$$\Rightarrow \quad \lim_{\theta\to 0}\frac{\sin n\theta}{\theta\cdot\frac{\sin\theta}{\theta}} = b_1 + 0 + 0 + 0 + ...$$

Other values becomes zero for higher powers of $\sin\theta$.

$$\Rightarrow \quad \frac{n\cdot 1}{1} = b_1$$

$\Rightarrow \quad b_1 = n.$ Therefore (B) is the Answer

16. $f(x) = \sin^4 x + \cos^4 x$

Differentiating w.r.t. x, we get

$f'(x) = 4\sin^3 4x \,.\, \cos x - 4\cos^3 x \,.\, \sin x$

$= 4 \sin x \cos x(\sin^2 x - \cos^2 x)$

$= 2 \,.\, \sin 2x(-\cos 2x)$

$= -\sin 4x$

Now, $f'(x) = 0$ if $\sin 4x < 0$

$\Rightarrow \quad \pi < 4x < 2\pi$

$\Rightarrow \quad \frac{\pi}{4} < x < \frac{\pi}{2}$ .....(i) and (A) is wrong

$\therefore \quad 0 < x < 3\pi/8$

Again (B) is the answer since (B) is proper subset of (i)

Again (C), $\frac{3\pi}{8} < x < \frac{5\pi}{8}$, is not the answer because C is not proper subset of (i).

Again (D) is not answer.

17. It is given that tan P/2 tan Q/2 are the roots of the quadratic equation

$ax^2 + bx + c = 0$ and $\angle R = \pi/2$

So $\quad \tan P/2 + \tan Q/2 = -b/a$

$\quad \tan P/2 \tan Q/2 = c/a$

Now, $\quad P + Q + R = 180°$

$\Rightarrow \quad P + Q = 90°$

$$\Rightarrow \quad \frac{P+Q}{2} = 45°$$

Taking tan of both sides $\tan\left(\frac{P+Q}{2}\right) = \tan 45°$

$$\Rightarrow \quad \frac{\tan P/2 + \tan Q/2}{1 - \tan P/2 \cdot \tan Q/2} = 1$$

$$\Rightarrow \quad \frac{-b/a}{1-c/a} = 1 \quad \Rightarrow \quad \frac{-b/a}{\frac{a-c}{a}} = 1 \quad \Rightarrow \quad \frac{-b}{a-c} = 1$$

$\Rightarrow \quad -b = a - c$

$\Rightarrow \quad a + b = c.$ Therefore, (A) is the answer

18. (B)

$$f_n(\theta) = \frac{\sin(\theta/2)}{\cos(\theta/2)}\left[\frac{2\cos^2(\theta/2)}{\cos\theta}\frac{2\cos^2\theta}{\cos 2\theta}\frac{2\cos^2 2\theta}{\cos 4\theta}....\right]$$

$$= \frac{\sin\theta}{\cos\theta}\left[\frac{2\cos^2\theta}{\cos 2\theta}\frac{2\cos^2 2\theta}{\cos 4\theta}....\right]$$

$$= \frac{\sin 2\theta}{\cos 2\theta}\left[\frac{2\cos^2 2\theta}{\cos 4\theta}....\right] = \tan 2^n\theta$$

$$f_3\left(\frac{\pi}{32}\right) = \tan 8\frac{\pi}{32} = \tan\frac{\pi}{4} = 1$$

19. (C)

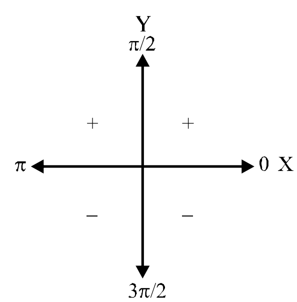
$f(\theta) = \sin\theta(\sin\theta + \sin 3\theta) = (\sin\theta + 3\sin\theta - 4\sin^3\theta)$

$= (4\sin\theta - 4\sin^3\theta)\sin\theta$

$= 4\sin^2\theta(1 - \sin^2\theta) = 4\sin^2\theta\cos^2\theta = (2\sin\theta\cos\theta)$

$= (\sin 2\theta)^2 \geq 0$

which is true for all θ.



20. (B)

$$\alpha + \beta = \frac{\pi}{2} \quad \Rightarrow \quad \alpha = \frac{\pi}{2} - \beta \quad \Rightarrow \quad \tan\alpha = \cot\beta$$

$\Rightarrow \quad \tan\alpha\tan\beta = 1$ .....(i)

Again, $\beta + \gamma \quad \Rightarrow \quad \gamma = \alpha - \beta$

$$\Rightarrow \quad \tan\gamma = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{\tan\alpha - \tan\beta}{2} \qquad \text{[Equ. (i)]}$$

$\Rightarrow \quad \tan\alpha = \tan\beta + 2\tan\gamma$

21. (A)

We are given that

$(\cot\alpha_1)(\cot\alpha_2).....(\cot\alpha_n) = 1$

$\Rightarrow$ $(\cos\alpha_1)(\cos\alpha_2)...(\cos\alpha_n) = (\sin\alpha_1)(\sin\alpha_2)...(\sin\alpha_n)$ .....(i)

Let $y = (\cos\alpha_1)(\cos\alpha_2)...(\cos\alpha_n)$ (to be maximum)

Squaring both sides, we get

$y^2 = (\cos^2\alpha_1)(\cos^2\alpha_2)...(\cos^2_n)$

$= \cos\alpha_1 \sin\alpha_1 \cos\alpha_2 \sin\alpha_2 ... \cos\alpha_n \sin\alpha_n$

$= \frac{1}{2^n}[\sin 2\alpha_1 \sin 2\alpha_2 .... \sin 2\alpha_n]$

As $0 \le \alpha_1, \alpha_2, ... \alpha_n \le \pi/2$

$\therefore$ $0 \le 2\alpha_1, 2\alpha_2, ... 2\alpha_2 \le \pi$

$\Rightarrow$ $0 \le \sin 2\alpha_1, \sin 2\alpha_2, ... \sin 2\alpha_n \le 1$

$\therefore$ $y^2 \le \frac{1}{2^n} 1$ $\Rightarrow$ $y \le \frac{1}{2^{n/2}}$

Therefore, the maximum value of y is $1/2^{n/2}$.

22. (B)

Given that $\sin\theta = 1/2$ and $\cos\phi = 1/3$, and θ and ϕ are acute angles.

$\therefore$ $\theta = \pi/6$ and $0 < \frac{1}{3} < \frac{1}{2}$

or $\cos\pi/2 < \cos\phi < \cos\pi/3$ or $\pi/3 < \phi < \pi/2$

$\therefore$ $\frac{\pi}{3} + \frac{\pi}{6} < \theta + \phi < \frac{\pi}{2} + \frac{\pi}{6}$ or $\frac{\pi}{2} < \theta + \phi < \frac{2\pi}{3}$

$\Rightarrow$ $\theta + \phi \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

23. (D)

Given that $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/e$

where $\alpha, \beta \in [-\pi, \pi]$

Now $\cos(\alpha - \beta) = 1$ $\Rightarrow$ $\alpha - \beta = 0$ or $\alpha = \beta$

$\therefore$ $\cos(\alpha + \beta) = 1/e$ $\Rightarrow$ $\cos 2\alpha = 1/e$

$\because$ $0 < 1/e < 1$ and $2\alpha \in [-2\pi, 2\pi]$

There will be two values of 2α satisfying $\cos 2\alpha = 1/e$ in $[0, 2\pi]$ and two in $[-2\pi, 0]$.

Therefore, there will be four values of α in $[-2\pi, 2\pi]$ and correspondingly four values of β.

Hence, there are four sets of (α, β).

24. (D)

$\theta \in \left(0, \frac{\pi}{4}\right)$ $\Rightarrow$ $\tan\theta < 1$ and $\cot\theta > 1$

Let $\tan\theta = 1$ and $\cot\theta = 1 + y$, where $x, y > 0$ and are very small, then

$t_1 = (1 - x)^{1-x}$, $t_2 = (1 - x)^{1+y}$, $t_3 = (1 + y)^{1-x}$, $t_4 = (1 + y)^{1+y}$

Clearly, $t_4 > t_3$ and $t_1 > t_2$. Also, $t_3 > t_1$.

Thus, $t_4 > t_3 > t_1 > t_2$.

25. (A, B)

$$\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$$

$$3\sin^4 x + 2(1 - \sin^2 x)^2 = \frac{6}{5}$$

$\Rightarrow$ $\sin^4 x - 20\sin^2 x + 4 = 0$

$\Rightarrow$ $\sin^2 x = \frac{2}{5}$

$\therefore$ $\cos^2 x = \frac{3}{5}$

$\therefore$ $\tan^2 x = \frac{2}{3}$

and $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

26. (2)

$$\frac{1}{4\cos^2\theta + 1 + \frac{3}{2}\sin 2\theta}$$

$\Rightarrow$ $\frac{1}{2[1 + \cos 2\theta] + 1 + \frac{3}{2}\sin 2\theta}$

lies between $\frac{1}{2}$ to $\frac{11}{2}$

$\therefore$ Maximum value is 2.

Minimum value of $1 + 4\cos^2\theta + 3\sin\theta\cos\theta$

$1 + \frac{4(1 + \cos 2\theta)}{2} + \frac{3}{2}\sin 2\theta$

$= 1 + 2 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$

$= 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$

$\therefore$ $= 3 - \sqrt{4 + \frac{9}{4}} = 3 - \frac{5}{2} = \frac{1}{2}$.

So maximum value of $\frac{1}{4\cos^2\theta + 1 + \frac{3}{2}\sin 2\theta}$ is 2.

27. (A, C, D)

$2\cos\theta(1 - \sin\phi) = \frac{2\sin^2\theta}{\sin\theta}\cos\phi - 1 = 2\sin\theta\cos\phi - 1$

$2\cos\theta - 2\cos\theta\sin\phi = 2\sin\theta\cos\phi - 1$

$2\cos\theta + 1 = 2\sin(\theta + \phi)$

$\tan(2\pi - \theta) > 0$ $\Rightarrow$ $\tan\theta < 0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$

$\Rightarrow$ $\theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$

$\Rightarrow$ $\frac{1}{2} < \sin(\theta + \phi) < 1$

$$\Rightarrow \quad 2\pi + \frac{\pi}{6} < \theta + \phi < \frac{5\pi}{6} + 2\pi$$

$$\Rightarrow \quad 2\pi + \frac{\pi}{6} - \theta_{max} < \phi < 2\pi + \frac{5\pi}{6} - \theta_{min}$$

$$\frac{\pi}{2} < \phi < \frac{4\pi}{3}$$

28. (A, B)

For $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Let $\cos 4\theta = 1/3$

$$\Rightarrow \quad \cos 2\theta = \pm\sqrt{\frac{1+\cos 4\theta}{2}} = \pm\sqrt{\frac{2}{3}}$$

$$f\left(\frac{1}{3}\right) = \frac{2}{2-\sec^2\theta} = \frac{2\cos^2\theta}{2\cos^2\theta - 1} = 1 + \frac{1}{\cos 2\theta}$$

$$f\left(\frac{1}{3}\right) = 1 - \sqrt{\frac{3}{2}} \text{ or } 1 + \sqrt{\frac{3}{2}}$$

29. (A)

Given expression is

$$\frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = \frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$= \frac{1}{\sin A - \cos A}\left\{\frac{\sin^3 A - \cos^3 A}{\cos A \sin A}\right\}$$

$$= \frac{\sin^2 A + \sin A \cos A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{1 + \sin A \cos A}{\sin A \cos A}$$

$= 1 + \sec A \operatorname{cosec} A$

30. (D)

$f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ where $x \in R$ and $k \geq 1$

Now, $f_4(x) - f_6(x)$

$$= \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x)$$

$$= \frac{1}{4}(1 - 2\sin^2 x \cdot \cos^2 x) - \frac{1}{6}(1 - 3\sin^2 x \cdot \cos^2 x)$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

31. (C)

$$\sum_{k=1}^{13} \frac{\sin\left[\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) - \left(\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right)\right]}{\sin\frac{\pi}{6}\left(\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)\sin\left(\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right)\right)}$$

$$= 2\sum_{k=1}^{13}\left(\cot\left(\frac{\pi}{4} + (k-1)\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)\right)$$

$$= 2\left(\cot\frac{\pi}{4} - \cot\left(\frac{\pi}{4} + \frac{13\pi}{6}\right)\right) = 2\left(1 - \cot\left(\frac{29\pi}{2}\right)\right)$$

$$= 2\left(1 - \cot\left(\frac{5\pi}{12}\right)\right) = 2(1 - (2 - \sqrt{3})) = 2(-1 + \sqrt{3})$$

$$= 2(\sqrt{3} - 1)$$

32. (D)

$5(\tan^2 x - \cos^2 x) = 2\cos^2 x + 9$

$5\left(t^2 - \frac{1}{1+t^2}\right) = 2\left(\frac{1-t^2}{1+t^2}\right) + 9$

$5(t^4 + t^2 - 1) = 2 - 2t^4 + 9 + 9t^4$

$5t^4 - 2t^2 - 16 = 0$

$5t^4 - 10t^2 + 8t^2 - 16 = 0$

$5t^2(t^2 - 2) + 8(t^2 - 2) = 0$

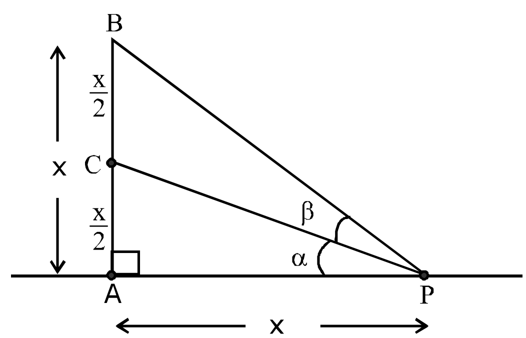
$(5t^2 + 8)(t^2 - 2) = 0$

$\tan^2 x = 2$

$\cos 2x = \frac{1-2}{1+2} = -\frac{1}{3}$

$\cos 4x = 2\cos^2 2x - 1 = -\frac{7}{9}$

33. (C)



$\tan\theta = \frac{1}{2}, \tan\alpha = \frac{1}{4}, \tan\beta = y$

$$\tan\theta = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\frac{1}{2} = \frac{\frac{1}{4} + y}{1 - \frac{y}{4}} \quad \Rightarrow \quad \frac{1}{2} = \frac{1+4y}{4-y}$$

$4 - y = 2 + 8y$

$\frac{2}{9} = y$

34. (BC)

$\cos\alpha = \left(\frac{1-a}{1+a}\right)$; $a = \tan^2\frac{\alpha}{2}$

$\cos\beta = \left(\frac{1-b}{1+b}\right)$; $b = \tan^2\frac{\beta}{2}$

$2\left(\left(\frac{1-b}{1+b}\right)-\left(\frac{1-a}{1+a}\right)\right)+\left(\left(\frac{1-a}{1+a}\right)\left(\frac{1-b}{1+b}\right)\right)=1$

$\Rightarrow 2((1-b)(1+a)-(1-a)(1+b))+(1-a)(1-b)$
$= (1+a)(1+b)$

$\Rightarrow 2(1+a-b-ab-(1+b-a-ab))+1-a-b+ab$
$= 1+a+b+ab$

$\Rightarrow 4(a-b) = 2(a+b)$

$\Rightarrow 2a-2b = a+b$

$\Rightarrow a = 3b$

$\Rightarrow \tan^2\frac{\alpha}{2} = 3\tan^2\frac{\beta}{2}$

$\Rightarrow \tan\frac{\alpha}{2} = \pm\sqrt{3}\tan\left(\frac{\beta}{2}\right)$

35. (0.5)

$\sqrt{3}\cos x + \frac{2b}{a}\sin x = \frac{c}{a}$

Now, $\sqrt{3}\cos\alpha + \frac{2b}{a}\sin\alpha = \frac{c}{a}$ .....(i)

$\Rightarrow \sqrt{3}\cos\beta + \frac{2b}{a}\sin\beta = \frac{c}{a}$ .....(ii)

$\Rightarrow \sqrt{3}[\cos\alpha - \cos\beta] + \frac{2b}{a}(\sin\alpha - \sin\beta) = 0$

$\Rightarrow \sqrt{3}\left[-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]$

$+ \frac{2b}{a}\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right] = 0$

$\Rightarrow -\sqrt{3}+2\sqrt{3}\cdot\frac{b}{a} = 0$

$\frac{b}{a} = \frac{1}{2} = 0.5$

MOCK TEST

SINGLE CORRECT CHOICE TYPE

1. (A) We have, $k_1 = \tan 27\theta - \tan\theta$

$= (\tan 27\theta - \tan 9\theta) + (\tan 9\theta - \tan 3\theta) + (\tan 3\theta - \tan\theta)$

Now, $\tan 3\theta - \tan\theta = \dfrac{\sin 2\theta}{\cos 3\theta\cos\theta} = \dfrac{2\sin\theta}{\cos 3\theta}$

Similarly, $\tan 9\theta - \tan 3\theta = \dfrac{2\sin 3\theta}{\cos 27\theta}$

and $\tan 27\theta - \tan 9\theta = \dfrac{2\sin 9\theta}{\cos 27\theta}$

$\therefore\quad k_1 = 2\left[\dfrac{\sin 9\theta}{\cos 27\theta} + \dfrac{\sin 3\theta}{\cos 9\theta} + \dfrac{\sin\theta}{\cos 3\theta}\right] = 2k_2$

2. (B) We have, $\dfrac{\sec^8\theta}{a} + \dfrac{\tan^8\theta}{b} = \dfrac{1}{a+b}$

or $a^2\sin^8\theta + ab + b^2 = ab(\cos^8\theta - \sin^8\theta)$

$= ab(1 - 2\sin^2\theta\cos^2\theta)\cos 2\theta$

$= ab(1 - 2\sin^2\theta)\left(1 - \dfrac{1}{2}\sin^2 2\theta\right)$

$\Rightarrow\quad a^2\sin^8\theta - 2ab\sin^4\theta + b^2 = -2ab\sin^2\theta$

$+ ab\sin^2\theta\sin^2 2\theta - \dfrac{ab}{2}\sin^2 2\theta - 2ab\sin^4\theta$

$\Rightarrow\quad (a\sin^4\theta - b)^2 = 2ab\sin^2\theta(-\sin^2\theta - 1$

$+ 2\sin^2\theta\cos^2\theta - \cos^2\theta)$

$\Rightarrow\quad 2ab\sin^2\theta(-2 + 2\sin^2\theta\cos^2\theta) \ge 0$

$\Rightarrow\quad 4ab\sin^2\theta(\sin^4\theta - \sin^2\theta + 1) \le 0$

$\Rightarrow\quad ab \le 0$

3. (A) $\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2(1+\cos\theta)}}}}$

(n number of 2's)

$\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{(2+2\cos\theta/2)}}}}$

((n – 1) number of 2's)

....

$= \sqrt{2+2\cos(\theta/2^{n-1})}$

$= \sqrt{2\{1+2\cos\theta^2(\theta/2^n)-1\}} = 2\cos(\theta/2^n)$

4. (B) $\sin\left(\dfrac{\pi}{14}\right)\sin\left(\dfrac{3\pi}{14}\right)\sin\left(\dfrac{5\pi}{14}\right)$

$= \cos\left(\dfrac{\pi}{2} - \dfrac{\pi}{14}\right)\cos\left(\dfrac{\pi}{2} - \dfrac{3\pi}{14}\right)\cos\left(\dfrac{\pi}{2} - \dfrac{5\pi}{14}\right)$

$= \cos\left(\dfrac{3\pi}{7}\right)\cos\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{\pi}{7}\right)$

$= \cos\left(\pi - \dfrac{4\pi}{7}\right)\cos\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{\pi}{7}\right)$

$= -\cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{4\pi}{7}\right)$

$\therefore\quad \cos\left(\dfrac{\pi}{7}\right)\cos\dfrac{2\pi}{7}\cos\left(\dfrac{10\pi}{7}\right) - \sin\left(\dfrac{\pi}{14}\right)\sin\left(\dfrac{3\pi}{14}\right)\sin\left(\dfrac{5\pi}{14}\right)$

$= \cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{2\pi}{7}\right)\cos\left(2\pi - \dfrac{10\pi}{7}\right)$

$+ \cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{4\pi}{7}\right)$

$= 2\cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{4\pi}{7}\right)$

$= \dfrac{2\sin\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{4\pi}{7}\right)}{\sin\left(\dfrac{\pi}{7}\right)}$

$= \dfrac{2\sin\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{4\pi}{7}\right)}{2\sin\left(\dfrac{\pi}{7}\right)}$

$= \dfrac{2\sin\left(\dfrac{4\pi}{7}\right)\cos\left(\dfrac{4\pi}{7}\right)}{2\times 2\sin\left(\dfrac{\pi}{7}\right)}$

$= \dfrac{\sin\left(\dfrac{8\pi}{7}\right)}{4\sin\left(\dfrac{\pi}{7}\right)} = \dfrac{\sin\left(\pi + \dfrac{\pi}{7}\right)}{4\sin\left(\dfrac{\pi}{7}\right)}$

$= -\dfrac{\sin\left(\dfrac{\pi}{4}\right)}{4\sin\left(\dfrac{\pi}{7}\right)} = -\dfrac{1}{4}$

5. (D) Let

$P = \log_{10} \tan 1° + \log_{10} \tan 2° + \log_{10} \tan 3° + ... + \log_{10} \tan 89°$

$= \log_{10} (\tan 1° \tan 2° \tan 3° .... \tan 45° .... \tan 89°)$

$= \tan_{10} 1 = 0$ $\quad (\because \tan 89° = \tan (90° - 1°) = \cot 1°)$

$\therefore \quad e^p = e^0 = 1$ $\quad (\because \tan 89° \tan 1° = 1)$

MULTIPLE CORRECT CHOICE TYPE

6. (C, D) $P_n(u)$ be a polynomial in u of degree n.

$\sin 2ux = 2\sin nx \cos nx$

$= \sin x \, P_{2n-1} (\cos x)$ or $\cos x \, P_{2n-1} (\sin x)$

7. (C, D) We have, $\dfrac{3 + \cot 76° \cot 16°}{\cot 76° + \cot 16°}$

$$= \frac{3\sin 76° \sin 16° + \cos 76° \cos 16°}{\sin(76° + 16°)}$$

$$= \frac{2\sin 76° \sin 16° + \cos(76° - 16°)}{\sin(92°)}$$

$$= \frac{\cos 60° - \cos 92° + \cos 60°}{\sin 92°}$$

$$= \frac{1 - \cos 92°}{\sin 92°}$$

$= \tan 46°$ [Alternate (C)]

$= \tan(90° - 44°) = \cot 44°$ [Alternate (D)]

8. (B, C, D) We have, $x = \dfrac{1 - \sin\phi}{\cos\phi}, \; y = \dfrac{1 + \cos\phi}{\sin\phi}$

Multiplying, we get $xy = \dfrac{(1 - \sin\phi)(1 + \cos\phi)}{\cos\phi \sin\phi}$

$\Rightarrow \quad 1 - \sin\phi + \cos\phi - \sin\phi \cos\phi$

$$xy + 1 = \frac{+\sin\phi\cos\phi}{\cos\phi\sin\phi}$$

$$= \frac{1 - \sin\phi + \cos\phi}{\cos\phi \sin\phi}$$

and $x - y = \dfrac{(1 - \sin\phi)\sin\phi - \cos\phi(1 + \cos\phi)}{\cos\phi\sin\phi}$

$$= \frac{\sin\phi - \sin^2\phi - \cos\phi - \cos^2\phi}{\cos\phi\sin\phi}$$

$$= \frac{\sin\phi - \cos - 1}{\cos\phi \sin\phi} = -(xy + 1)$$

Thus, $xy + x - y + 1 = 0$

$\Rightarrow \quad x = \dfrac{y - 1}{y + 1}$ and $y = \dfrac{1 + x}{1 - x}$

9. (A, B, C, D) $\sin^2 A + \sin^2 B + \sin^2 C$

$$= \frac{1}{2}(1 - \cos 2A) + \frac{1}{2}(1 - \cos 2B) + \sin^2 C$$

$$= 1 - \frac{1}{2}(\cos 2A + \cos 2B) + \sin^2 C$$

$$= 1 - \frac{1}{2}\{2\cos(A + B) \cdot \cos(A -)B\} + \sin^2 C$$

$= 1 + \cos C \cdot \cos (A - B) + 1 - \cos^2 C$

$= 2 - \cos^2 C + \cos C \cdot \cos (A - B)$

$\therefore \quad \sin^2 A + \sin^2 B + \sin^2 C \le 2 - \cos^2 C + \cos C$ .....(i)

$= 2 - (\cos^2 C - \cos C)$

$$= 2 - \left(\cos^2 C - \cos C + \frac{1}{4}\right) + \frac{1}{4}$$

$$= \frac{9}{4} - \left(\cos C - \frac{1}{2}\right)^2$$

$\therefore \quad \sin^2 A + \sin^2 B + \sin^2 C \le \dfrac{9}{4}$

Now, for positive quantities AM $\ge$ GM

$$\therefore \quad \frac{\sin^2 A + \sin^2 B + \sin^2 C}{3} \ge \sqrt[3]{\sin^2 A \sin^2 B \sin^2 C}$$

$\therefore$ From Equ.(i), $\dfrac{9/4}{3} \ge \dfrac{\sin^2 A + \sin^2 B + \sin^3 C}{3}$

$\ge (\sin A \sin B \sin C)^{2/3}$

$\dfrac{3}{4} \ge (\sin A \sin B \sin C)^{2/3}$

or $\sin A \sin B \sin C \ge \left(\dfrac{3}{4}\right)^{3/2}$, i.e., $\dfrac{3\sqrt{3}}{8}$

Also in ΔABC, $\sin A > 0$, $\sin B > 0$, $\sin C > 0$ and form Equ. (i)

$\sin^2 A + \sin^2 B \le 1 + \cos C$

10. (A, B, C) $\dfrac{x}{y} = \dfrac{\cos A}{\cos B}$

$\Rightarrow \quad \dfrac{x}{\cos A} = \dfrac{y}{\cos B} = \lambda$ (say)

$\therefore \quad x = \lambda \cos A, \; y = \lambda \cos B$

Alternate (A)

$$\text{RHS} = \frac{x \tan A + y \tan B}{x + y}$$

$$= \frac{\lambda \cos A \tan A + \lambda \cos B \tan B}{\lambda \cos A + \lambda \cos B}$$

$$= \left(\frac{\sin A + \sin B}{\cos A + \cos B}\right)$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$

$$= \tan\left(\frac{A+B}{2}\right) = \text{LHS}$$

Alternate (B)

$$\text{RHS} = \frac{x\tan A - y\tan B}{x+y}$$

$$= \frac{\lambda\cos A\tan A - \lambda\cos B\tan B}{\lambda\cos A + \lambda\cos B}$$

$$= \frac{\sin A - \sin B}{\cos A + \cos B}$$

$$= \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$$

$$= \tan\left(\frac{A-B}{2}\right) = \text{LHS}$$

Alternate (C)

$$\text{RHS} = \frac{y\sin A + x\sin B}{y\sin A - x\sin B}$$

$$= \frac{\lambda\cos B\sin A + \lambda\cos A\sin B}{\lambda\cos B\sin A - \lambda\cos A\sin B}$$

$$= \frac{\sin(A+B)}{\sin(A-B)} = \text{LHS}$$

Alternate (D)

LHS = $x\cos A + y\cos B = \lambda\cos^2 A + \lambda\cos^2 B \neq 0$

SUBJECTIVE

11. (5) Given, expression

$$E = \cos\frac{\pi}{7} + \cos^2\frac{\pi}{7} - 2\cos^3\frac{\pi}{7}$$

$$= \cos\frac{\pi}{7}\left[1 + \cos\frac{\pi}{7} - 2\cdot\cos^2\frac{\pi}{7}\right]$$

$$= \cos\frac{\pi}{7}\left[\cos\frac{\pi}{7} - (2\cos^2\frac{\pi}{7} - 1)\right]$$

$$= \cos\frac{\pi}{7}\left[\cos\frac{\pi}{7} - \cos\frac{2\pi}{7}\right], \quad (\text{As } 2\cos^2\theta - 1 = \cos 2\theta)$$

$$= \cos\frac{\pi}{7}.\left[\cos\frac{\pi}{7} + \cos\frac{5\pi}{7}\right] \quad (\text{As, } \frac{2\pi}{7} + \frac{5\pi}{7} = \pi)$$

$$\Rightarrow \cos\frac{2\pi}{7} = \cos\left(\pi - \frac{5\pi}{7}\right) \quad \Rightarrow \cos\frac{2\pi}{7} = -\cos\frac{5\pi}{7}$$

$$= \cos\frac{\pi}{7}\left[2\cdot\cos\frac{3\pi}{7}\cdot\cos\frac{2\pi}{7}\right]$$

$$= 2\cdot\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}$$

$$= -2\cdot\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{3\pi}{7}$$

$$= \frac{-8\cdot\sin\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}}{4\cdot\sin\frac{\pi}{7}}$$

$$(\text{As, } \cos\frac{3\pi}{7} = -\frac{\cos 4\pi}{7})$$

$$= -\sin\frac{8\pi}{4\cdot\sin\frac{\pi}{7}} = \frac{-\sin\left(\pi + \frac{\pi}{7}\right)}{\sin\cdot\sin\frac{\pi}{7}} = \frac{1}{4} = \frac{p}{q}$$

So, $p = 1$ & $q = 4$

Hence, $(p+q) = 1 + 4 = 5$

12. (2) As, $\frac{\cos 3nA}{\cos nA} = \frac{4\cos^3 nA - 3\cos nA}{\cos nA}$

$= (4\cos^2 nA - 3) = 2(1 + \cos 2nA) - 3$

$= (2\cos 2nA - 1)$

$$\Rightarrow \left(1 + \frac{\cos 3nA}{\cos nA}\right) = 2\cos 2nA$$

$\therefore$ The given equation becomes

$2[\cos 2A + (\cos 4A + \cos 6A) + \cos 8A] = 0$

$$\Rightarrow \frac{\cos 5A\cdot\sin 4A}{\sin A} = 0$$

As $A \in \left(0, \frac{\pi}{4}\right)$, so $4A \in (0, \pi)$ and $5A \in \left(0, \frac{\pi}{4}\right)$

$\therefore$ sin 4A ≠ 0, so cos 5A = 0

$\Rightarrow$ $5A = \frac{\pi}{2}$ $\Rightarrow$ $A = \frac{\pi}{10} = 18°$

Hence, (cosec A – sec 2A) = (cosec 18° – sec 36°)

$= \frac{1}{\sin 18°} - \frac{1}{\cos 36°}$

$= \left(\frac{4}{\sqrt{5}-1} - \frac{4}{\sqrt{5}+1}\right) = (\sqrt{5}+1) - (\sqrt{5}-1)$

$= 1 + 1 = 2$

13. (3) We have, $(4\cos^2 40° - 3)(3 - 4\sin^2 40°)$

$= \frac{(4\cos^3 40° - 3\cos 40°)(3\sin 40° - 4\cdot\sin^3 40°)}{\cos 40° \cdot \sin 40°}$

$= \frac{2\cdot\sin 120°\cdot\cos 120°}{2\cdot\sin 40°\cdot\cos 40°} = \frac{\sin 240°}{\sin 80°} = \frac{\sin(3\times 80°)}{\sin 80°}$

$= \frac{3\sin 80° - 4\cdot\sin^3 80°}{\sin 80°}$

$= (3 - 4\sin^2 80°) = (3 - 4\cos^2 10°) = 3 - 2(1 + \cos 20°)$

$= (1 - 2\cos 20°) = (a + b\cos 20°)$ (Given)

$\therefore$ a = 1, b = –2

Hence, (a – b) = 1 – (–2) = 1 + 2 = 3

14. (4) We have,

$\alpha = 4\sin^2 10° + 2(2\cdot\sin 50°\cdot\cos 20°)\cdot\sin 50° + \cos 80°$

$= 4\sin^2 10° + 2\,[\sin 70° + \frac{1}{2}]\cdot\sin 50° + \cos 80°$

$= 2(2\cdot\sin^2 10°) + (2\cdot\sin 70°\cdot\sin 50°) + \sin 50° + \cos 80°$

$= 2(1 - \cos 20°) + (\cos 20° - \cos 120°) + \sin 50° + \cos 80°$

$= 2 - 2\cos 20° + \cos 20° + \frac{1}{2} + \sin 50° + \cos 80°$

$= \frac{5}{2} + [2\cdot\sin 50°\cdot\sin(-30°)] + \sin 50°$

$= \frac{5}{2} - 2\times\frac{1}{2}\times\sin 50° + \sin 50°$

Hence, $\alpha = \frac{5}{2}$

Similarly, $\beta = \cos^2 36° + \cos^2 24° + \cos^2 96°$

$= \cos^2 36° + \cos^2 24° + \sin^2 6°$

(As $\cos^2 96° = \cos^2(90° + 6°) = \sin^2 6°$)

$\therefore$ $\beta = \left(\frac{1+\cos 72°}{2}\right) + \left(\frac{1+\cos 48°}{2}\right) + \left(\frac{1-\cos 12°}{2}\right)$

$= \frac{3}{2} + \frac{1}{2}[\cos 72° + \cos 48°] - \frac{1}{2}\cdot\cos 12°$

$= \frac{3}{2} + \frac{1}{2}[2\cdot\cos 60°\cdot\cos 12°] - \frac{1}{2}\cdot\cos 12°$

$\therefore$ $\beta = \frac{3}{2} + \frac{1}{2}\cdot\cos 12° - \frac{1}{2}\cdot\cos 12° = \frac{3}{2}$ $\Rightarrow$ $\beta = \frac{3}{2}$

Hence, $(\alpha + \beta) = \frac{5}{2} + \frac{3}{2} = \frac{8}{2} = 4$

15. (24) We know that

$\sin\theta\cdot\sin(60° - \theta)\cdot\sin(60° + \theta) = \frac{1}{4}\sin 3\theta$ .....(i)

Put θ = 25° in (i), we get

$\sin 25°\cdot\sin(60° - 25°)\cdot\sin(60° + 25°)$

$= \frac{1}{4}.\sin(3\times 25°) = \frac{1}{4}\sin 75°$

$= \frac{1}{4}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{(\sqrt{3}+1)\times\sqrt{2}}{8\sqrt{2}\times\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{16} = \frac{\sqrt{a}+\sqrt{b}}{c}$

$\therefore$ a = b, c = 2 & c = 16

Hence, (a + b + c) = 6 + 2 + 16 = 24

INTEGER

16. (1)

As, $\sec\theta + \sec(\pi - \theta) = 0$

So, $f = \sec 0 + \left(\sec\frac{\pi}{7} + \sec\frac{6\pi}{7}\right) + \left(\sec\frac{2\pi}{7} + \sec\frac{5\pi}{7}\right) + \left(\sec\frac{3\pi}{7} + \sec\frac{4\pi}{7}\right)$

$\therefore$ f = 1

Also, $\cot\theta + \cot(180° - \theta) = 0$

So, c = (cot 1° + cot 179°) + (cot 2° + cot 178°) + ... + (cot 89° + cot 91°) + cot 90°

$\therefore$ c = 0

Hence, $(c^2 + f^2) = 1$

17. (1) $\frac{\sin^2 34° - \sin^2 11°}{\sin 34°\cos 34° - \sin 11°\cos 11°}$

$= \frac{2[\sin 45°\cdot\sin 23°]}{\sin 68° - \sin 22°}$

$= \frac{2\sin 45°\sin 23°}{2\cos 45°\sin 23°} = \tan 45° = 1$

18. (1)

$$\prod_{k=0}^{5} 2\cos(2^k A) = 2^6 \cos A \cos 2A \cos^2 A \,.....\, \cos 2^5 A$$

$$= \frac{\sin(2^6 A)}{\sin A} = \frac{\sin\left(\frac{64\pi}{65}\right)}{\sin\left(\frac{\pi}{65}\right)} = 1$$

19. (1)

$T_r = \tan r° \tan(r+1)° + 1 - 1$

$$T_r = \frac{\sin r° \sin(r+1)° \cos r° \cos(r+1)°}{\cos r° \cos(r+1)°} - 1$$

$$T_r = \frac{\cos 1°}{\cos r° \cos(r+1)°} - 1$$

$$T_r = \frac{\cos 1°}{\sin 1°}\left(\frac{\sin((r+1)° - r°)}{\cos r° \cos(r+1)°}\right) - 1$$

$T_r = \cot 1°(\tan(r+1)° - \tan r°) - 1$

$T_1 + T_2 + ... + T_{18} = \cot^2 1° - 89.$

20. (6)

$y = (\sin x - 1)^2 - \cos^2 x$

$y = (\sin x - 1)^2 - (1 - \sin^2 x)$

$$y = 2\left(\left(\sin x - \frac{1}{2}\right)^2 - \frac{1}{4}\right)$$

$y_{max} = 4$ when $\sin x = -1$

$y_{max} = -\frac{1}{2}$ when $\sin x = \frac{1}{2}$

$\therefore\ M = 4\ ;\ m = -\frac{1}{2}$